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On Partly Gentle Perturbations III.*

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1. INTRODUCTION

In part I of this paper [A.13] a set of criteria was formulated for the unitary equivalence of two self-adjoint operators acting in an abstract Hilbert space. These criteria were modeled after the gentleness criteria of Friedrichs and were called partial gentleness criteria.

In part II of this paper [A.14], it was illustrated that these criteria can be verified for a class of potential perturbations of, Δ, the Laplacian. Potentials of this class were required to satisfy a version of the Povzner-Ikebe condition.

In the present part III of this paper, we generalize the Povzner-Ikebe condition. Our aim is to improve the behavior of the potential near infinity. We also allow the same local behavior as Povzner [A.1] and Ikebe [A.2] did. This leads to a technical difficulty. If we would restrict this local behavior as we did in part II then the proofs would be considerably shorter. However, we wish to illustrate that the partial gentleness criteria can be verified for such potentials.

In Section 2 we first introduce two conditions on the potential. Condition I is a restriction on the local behavior. Such a condition has been used in connection with the study of the essential spectra by Schechter [B.5] and elsewhere [B.4]. Condition II is both a local and a global restriction on the potential. In fact according to the Appendix, Condition II implies Condition I. Our main requirement on the potential p is that it can be written in the form $p = p_1 + p_2$, where p_1 satisfies Condition I and p_2 satisfies Condition II. Roughly speaking, in this decomposition, p_1 is the locally bad part of p. In Theorem 2.1 we show that for such potentials, under mild continuity and other assumptions, the continuous part of the Friedrichs extension of $-\Delta + M(p)$ is unitarily equivalent to $-\Delta$. In case of dimension $d = 1$, Theorem 2.1 gives a theorem of Titchmarsh [A.17] and in case of dimension

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$d = 3$ it gives a generalization of the theorem of Povzner [A.1] and Ikebe [A.2]. Let us remark that the continuity assumption of Theorem 2.1 could be weakened. For example it can be replaced by the assumption that the potential is small. However, we have not been able to remove it completely. According to a verbal communication of Kuroda this question will be studied in his forthcoming paper [A.16].

In Section 3 we formulate the partial gentleness criteria and a previous theorem on partly gently perturbations. This is done in Theorem 3.1. These criteria consist of two parts. We refer to the first part as partial gentleness conditions and to the second part as additional conditions. These conditions are stated with reference to a given Banach space $\mathfrak{B}$ which is not unique. The construction of such a space makes essential use of the specific properties of the given perturbation problem.

In Section 4 we verify the first conclusion of Theorem 2.1. That is, we show that the operator $-\Delta + M(p)$ is bounded from below. Hence it does admit a Friedrichs extension that we denote by $(-\Delta + M(p))_F$.

In Section 5 we formulate estimates for the operator valued Green's function of the Laplacian. Roughly speaking one obtains this operator valued function from the usual Green's function by freezing the radial variables and letting it act only on the angular variables. Crude estimates for the norm of this operator were formulated in Theorem 4.1 of part II. At present we make essential use of the fact that the Laplacian admits a family of reducing subspaces such that on each of these it acts like an ordinary differential operator. We use this fact and a uniform estimate for Bessel functions, formulated elsewhere [B.11], to obtain estimates for the operator valued Green's function. This leads to Theorem 5.1.

In Section 6 we derive Theorem 2.1 from the abstract Theorem 3.1. Specifically for the unperturbed operator we take $-\Delta$ and for the perturbed operator we take $(-\Delta + M(p))_F$. Then introduce a partial gentleness norm with reference to which the assumptions of Theorem 3.1 hold for the pair $(-\Delta, (-\Delta + M(p))_F)$. This partial gentleness norm consists of three terms. The first term corresponds to the potential p_1 in the decomposition $p = p_1 + p_2$ of Section 2. This term is a norm and it is similar to a norm used elsewhere [A.7]. The second term corresponds to the potential p_2. It is also a norm and it is analogous but more general than the partial gentleness norm introduced in part II. The third term is due, so to speak, to the interaction between the potentials p_1 and p_2. Since our partial gentleness norm consists of three terms it is involved to work with it. In case p_1 is zero these three terms reduce to a single one. Hence according to Section 2 the appearance of the other two terms is due to the local behavior of p. This is the previously mentioned technical difficulty. To verify the partial gentleness assumptions of Theorem 3.1 with reference to this norm we need the estimates of Theorem 5.1 and some well known properties of the Green's function of Δ. To verify the additional assumptions of Theorem 3.1 we need a deep result of Kato [B.1] for dimensions $d \geqslant 2$ and a lemma of Titchmarsh [B.7.g] for $d = 1$. It is the application of this Kato result where we use the continuity assumption on p.

In conclusion let us remark that our generalization of the Povzner-Ikebe theorem overlaps with the recent results of Kato [A.8], Kuroda [A.16] and Krohn [A.15].

2. Formulation of an Extended Povzner–Ikebe Theorem

Let $\mathscr{E}_d$ denote the real Euclidean space of dimension d, and let $\ddot{\mathfrak{C}}_\infty$ denote the class of infinitely differentiable complex valued functions on $\mathscr{E}_d$ whose support is bounded and does not contain the origin. As is well known [B.9] the Laplacian is essentially self-adjoint on $\ddot{\mathfrak{C}}_\infty$ and we denote the closure by Δ. For a given measurable function p, to which we shall refer as a potential, we denote by $M(p)$ the operator of multiplication by p, i.e. set

$$M(p)f(x) = p(x)f(x).$$

In case of dimension $d = 3$ Povzner [A.1] formulated a condition on the potential p, which ensures that $-\Delta + M(p)$ is essentially self-adjoint on $\ddot{\mathfrak{C}}_\infty$ and that the continuous part of its closure is unitarily equivalent to $-\Delta$. His result was extended by Ikebe [A.2] who showed that it suffices to assume the following:

Condition P-I. *The potential p is real, square integrable over all of $\mathscr{E}_3$ and there is a positive number ϵ such that*

$$p(x) = O\left(\frac{1}{|x|}\right)^{2+\epsilon} \qquad at \qquad |x| = \infty.$$

Furthermore p is Hölder continuous with the exception of finitely many points.

Our aim is to generalize Condition P-I, in particular to generalize the growth condition at infinity. To describe such a generalization we first introduce some notations. For a given potential p we set

$$p^*(\xi) = \sup_{|x|=\xi} |p(x)|, \tag{2.1}$$

and if a pair of positive numbers (d, δ) is also given, we set

$$I_{d,\delta}(p) = \sup_{|x|} \left\{ \begin{array}{ll} \int_{|x-y|<\delta} | p(y) | \, dy, & d = 1 \\ \int_{|x-y|<\delta} | p(y) | \log \frac{1}{| x - y |} \, dy, & d = 2 \\ \int_{|x-y|<\delta} \left(\frac{1}{| x - y |} \right)^{d-2} | p(y) | \, dy, & d \geqslant 3 \end{array} \right\}. \tag{2.2}$$

With the aid of these notations we formulate:

CONDITION I. *The function p_1 is square integrable on any compact set which does not contain the origin, and for each positive number δ,*

$$I_{d,\delta}(p_1) < \infty \qquad \textit{and} \qquad \lim_{\delta=0} I_{d,\delta}(p_1) = 0.$$

CONDITION II. *The function p_2 is square integrable on any compact set which does not contain the origin and*

$$\left\{ \begin{array}{ll} \int_0^\infty p_2^*(\xi) \, d\xi, & d = 1 \\ \int_0^1 \xi^{1/2} p_2^*(\xi) \, d\xi + \int_1^\infty \xi^{1/3} p_2^*(\xi) \, d\xi, & d = 2 \\ \int_0^1 \xi p_2^*(\xi) \, d\xi + \int_1^\infty \xi^{1/3} p_2^*(\xi) \, d\xi, & d \geqslant 3 \end{array} \right\} < \infty.$$

It is nearly evident that for $d \geqslant 2$ Condition I does not imply Condition II, even if we assume that p_1 has bounded support. According to the Appendix, Condition II does imply Condition I. Incidentally, note that for $d = 3$, Condition I is implied by the square integrability assumption of Condition P-I.

We shall also need a condition introduced by Kato [B.1] in connection with his investigations of the reduced wave equation. It reads as follows:

CONDITON K. *There is a number ξ_0 such that*

$$\int_{\xi_0}^\infty p^*(\xi) \, d\xi < \infty \qquad \textit{and} \qquad \lim_{\xi=\infty} \xi p^*(\xi) = 0.$$

With the aid of these conditions we formulate the following extension of the theorems of Povzner-Ikebe [B.2] and Titchmarsh [B.7.h] [A.17].

THEOREM 2.1. *Let the real potential p be of the form $p = p_1 + p_2$, where p_1 satisfies Condition* I *and p_2 satisfies Condition* II. *Suppose that p_1 has bounded support and that the operator $M(p_1)$ is Δ-compact. Suppose further that for dimensions $d \geqslant 2$ the potential p is locally Hölder continuous with the exception of finitely many points and that it satisfies Condition* K. *Then the operator $-\Delta + M(p)$ on $\ddot{\mathfrak{C}}_\infty$ is bounded from below. The continuous part of its Friedrichs extension is unitarily equivalent to $-\Delta$.*

To see that this theorem extends the Povzner-Ikebe theorem replace the growth requirement in Condition P-I by the following more general one

$$p(x) = O\left(\frac{1}{|x|}\right)^{1+\frac{1}{2}+\epsilon} \qquad at \qquad |x| = \infty.$$

For brevity assume that this holds in the neighborhood $|x| > 1$. Suppose that p satisfies such a generalized Condition P-I and set

$$p_1(x) = \begin{cases} p(x) & |x| < 1 \\ 0 & |x| > 0 \end{cases} \qquad and \qquad p_2(x) = p(x) - p_1(x).$$

Then Theorem 2.1 applies to such a potential p and according to general considerations [B.4], [B.5], the operator $-\Delta + M(p)$ is essentially self-adjoint on $\ddot{\mathfrak{C}}_\infty$. Hence the conclusion of the Povzner-Ikebe theorem follows.

Let us remark that most likely, the assumption that p_1 has bounded support and $M(p_1)$ is Δ-compact can be weakened.

3. THE PREVIOUS THEOREM ON PARTLY GENTLE PERTURBATIONS

Let the operators A_0 and A_1 act in an abstract Hilbert space $\mathfrak{H}$ and assume that they are self-adjoint on the given domains $\mathfrak{D}(A_0)$ and $\mathfrak{D}(A_1)$. These domains need not be equal but we assume that their intersection is dense and we set

$$V = A_1 - A_0 \qquad \text{on} \qquad \mathfrak{D}(A_0) \cap \mathfrak{D}(A_1).$$

Let $\mathscr{I}$ be a given bounded interval and let $E_0(\mathscr{I})$ and $E_1(\mathscr{I})$ denote the spectral projector of these operators over $\mathscr{I}$. We denote by $A_0(\mathscr{I})$ and $A_1(\mathscr{I})$ the part of these operators over $\mathscr{I}$, that is their restriction to $E_0(\mathscr{I})\,\mathfrak{H}$ and $E_1(\mathscr{I})\,\mathfrak{H}$.

Next let $\mathfrak{B}$ be a Banach space such that both $\mathfrak{B}$ and $\mathfrak{H}$ can be embedded in a metric space $\mathfrak{M}$. We assume that this embedding is such that, $\mathfrak{B} \cap \mathfrak{H}$ is dense in $V(\mathfrak{D}(A_0) \cap \mathfrak{D}(A_1))$ with reference to the $\mathfrak{H}$-norm and in $\mathfrak{B}$ with reference to the $\mathfrak{B}$-norm. We assume further that V considered as a mapping of $\mathfrak{D}(A_0) \cap \mathfrak{D}(A_1)$ into $\mathfrak{M}$ is continuous with reference to the $\mathfrak{M}$-metric, and hence it can be extended to all of $\mathfrak{H}$. In applications the abstract Hilbert space $\mathfrak{H}$ is an $\mathfrak{L}_2$ space and for $\mathfrak{M}$ we choose the space of measurable functions. Then these requirements are practically no restrictions.

As is well known [B.7.d], the unperturbed resolvent set, $\rho(A_0)$, contains the points of the open upper or lower half planes that we denote by $\mu_\pm$. For a given interval $\mathscr{I}$ and angle α between 0 and $\pi/2$ we define the regions $\mathscr{R}_\pm(\mathscr{I})$ by the relations

$$\mathscr{R}_\pm(\mathscr{I}) = E[\mu_\pm : \operatorname{Re} \mu_\pm \in \mathscr{I}, 0 < | \arg \mu_\pm | < \alpha]. \tag{3.1$_\pm$}$$

If $\mathscr{I}$ is in the spectrum of A_0 then the resolvent $R_0(\mu_\pm) = (\mu_\pm - A_0)^{-1}$ can not be continued onto $\mathscr{I}$ as a bounded operator on $\mathfrak{H}$. Accordingly we speak about the two resolvents in the two disjoint regions $\mathscr{R}_\pm(\mathscr{I})$.

Now we formulate the previously mentioned criteria which allow us to continue the perturbed resolvent onto $\mathscr{I}$ as a form on $\mathfrak{B} \times \mathfrak{B}$. To describe this in more specific terms we introduce a convention. We say that the operators $R_0(\mu_\pm)$ on $\mathfrak{H}$ determine bounded forms on $\mathfrak{B} \times \mathfrak{B}$, if the forms

$$[R_0(\mu_\pm)]_{\mathfrak{B}}\,(f, g) = (f, R_0(\mu_\pm)\, g) \qquad \text{on} \qquad (\mathfrak{B} \cap \mathfrak{H}) \times (\mathfrak{B} \cap \mathfrak{H})$$

are bounded with reference the $\mathfrak{B}$-norm. We denote the closure of these forms which are defined on all of $\mathfrak{B} \times \mathfrak{B}$, by the same symbols $[R_0(\mu_\pm)]_{\mathfrak{B}}$. We say that the operators $VR_0(\mu_\pm)$ in $\mathfrak{H}$ determine bounded operators on $\mathfrak{B}$, if

$$VR_0(\mu_\pm)\,(\mathfrak{B} \cap \mathfrak{H}) \subset \mathfrak{B}$$

and this mapping is bounded with reference the $\mathfrak{B}$-norm. Note that in general $VR_0(\mu_\pm)$ maps $\mathfrak{H}$ into $\mathfrak{M}$.

CONDITION $G_1(\mathscr{I})$. *For each $\mu_\pm$ in the open regions $\mathscr{R}_\pm(\mathscr{I})$ the operators $R_0(\mu_\pm)$ on $\mathfrak{H}$ determine bounded forms on $\mathfrak{B} \times \mathfrak{B}$ and the forms $[R_0(\mu_\pm)]_{\mathfrak{B}}$ admit weakly continuous extension onto the closures $\overline{\mathscr{R}_\pm(\mathscr{I})}$. Furthermore the norms of these forms remain bounded independently of $\mu_\pm$.*

CONDITION $G_2(\mathscr{I})$. *For each $\mu_\pm$ in the open regions $\mathscr{R}_\pm(\mathscr{I})$ the operators $VR_0(\mu_\pm)$ in $\mathfrak{H}$ determine bounded operators on $\mathfrak{B}$. These operators, $(VR_0(\mu_\pm))_{\mathfrak{B}}$, depend continuously in norm on $\mu_\pm$ and admit continuous extension onto the $\mathscr{R}_\pm(\mathscr{I})$.*

Actually in these two conditions it would be sufficient to assume the existence of the radial limit only, but we shall not be concerned with this fact. Next we assume that the operator V can be approximated in the following manner.

CONDITION $G_3(\mathscr{I})$. *There is a sequence of operators, $\{V_k\}$, such that for each k and $\mu_\pm$ in the open region $\mathscr{R}_\pm(\mathscr{I})$, the operators $V_k R_0(\mu_\pm)$ are defined on all of $\mathfrak{H}$ and are bounded. The pair $(A_0, A_0 + V_k)$ satisfies Conditions $G_1(\mathscr{I})$ and $G_2(\mathscr{I})$. Furthermore*

$$\lim_{k=\infty} \| (VR_0(\mu_\pm))_{\mathfrak{B}} - (V_k R_0(\mu_\pm))_{\mathfrak{B}} \| = 0.$$

Note that if V is A_0 bounded with reference the $\mathfrak{H}$-norm then we can set $V_k = V$. That is, in this case, Conditions $G_1(\mathscr{I})$ and $G_2(\mathscr{I})$ imply Condition $G_3(\mathscr{I})$. We refer to these three conditions by saying that the pair of operators (A_0, A_1) is gentle over the interval $\mathscr{I}$, in short partly gentle. Next we state the two additional conditions.

CONDITION $A_1(\mathscr{I})$. *For every ω in the closed and bounded interval $\mathscr{I}$, the operators $(1 - VR_0(\omega_\pm))_{\mathfrak{B}}$ are invertible. That is, they admit bounded inverses defined on all of $\mathfrak{B}$.*

CONDITION $A_2(\mathscr{I})$. *For each $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ the operators $R_1(\mu_\pm)$ on $\mathfrak{H}$ determine bounded forms on $\mathfrak{B} \times \mathfrak{B}$. These forms are related to the unperturbed resolvent via the second resolvent equation,*

$$[R_1(\mu_\pm)]_{\mathfrak{B}} - [R_0(\mu_\pm)]_{\mathfrak{B}} = [R_1(\mu_\pm)]_{\mathfrak{B}}\, (VR_0(\mu_\pm))_{\mathfrak{B}}\,.$$

An elementary argument shows [A.13] that if the operator V is A_0-bounded with reference the $\mathfrak{H}$-norm then the gentleness conditions and Condition $A_1(\mathscr{I})$ imply Condition $A_2(\mathscr{I})$.

After these preparations we formulate a theorem on such perturbations which was established elsewhere [A.13].

THEOREM 3.1. *Suppose that the pair of operators (A_0, A_1) is gentle over the closed and bounded interval $\mathscr{I}$. Suppose further that Conditions $A_1(\mathscr{I})$ and $A_2(\mathscr{I})$ hold. Then $A_0(\mathscr{I})$ and $A_1(\mathscr{I})$, the part of these operators over $\mathscr{I}$, are unitarily equivalent.*

4. THE FRIEDRICHS EXTENSION OF $-\Delta + M(p)$

In the present short section we verify the first conclusion of Theorem 2.1. That is, we show that $-\Delta + M(p)$ is bounded from below on $\ddot{\mathfrak{C}}_\infty$. Hence it does admit a Friedrichs extension [B.7.e] that we denote by $(-\Delta + M(p))_F$.

According to the Appendix, Condition II implies Condition I. At the same time we see from estimate (A.1) that

$$\lim_{|x|=\infty} \int_{|x-y|<1} | p_2(y) | \, dy = 0. \tag{4.1}$$

Hence the potential $p = p_1 + p_2$ satisfies Condition I and relation (4.1). According to general considerations used by Schechter [B.5] and elsewhere [B.4], this two facts imply that the operator

$$M(| p^{1/2} | (1 - \Delta)^{-1} M(| p^{1/2} |) \quad \text{is compact.}$$

Since the operators $M(| p^{1/2} |) (1 - \Delta)^{-1/2}$ and $(1 - \Delta)^{-1/2} M(| p^{1/2} |)$ are adjoint to each other we see that, also

$$(1 - \Delta)^{-1/2} M(p) (1 - \Delta)^{-1/2} \quad \text{is compact.} \tag{4.2}$$

It is a general operator-theoretic fact that the compactness of this operator ensures the semi-boundedness of $-\Delta + M(p)$. This is already implicit in the work of Birman [B.2] and for convenience we state and prove it in the lemma that follows.

LEMMA 4.1. *Let the operator V be symmetric on $\mathfrak{D}((1 - \Delta)^{1/2})$ and suppose that*

$$(1 - \Delta)^{-1/2} V(1 - \Delta)^{-1/2} \text{ is compact.} \tag{4.3}$$

Then the sesquilinear form of the operator $1 - \Delta + V$ is bounded from below on $\mathfrak{D}((1 - \Delta)^{1/2}) \times \mathfrak{D}((1 - \Delta)^{1/2})$.

To verify this lemma note that

$$1 - \Delta + V = (1 - \Delta)^{1/2} [1 + (1 - \Delta)^{-1/2} V(1 - \Delta)^{-1/2}] (1 - \Delta)^{1/2} \quad \text{on} \quad \mathfrak{D}((1 - \Delta)^{1/2}). \tag{4.4}$$

Let $\mathfrak{F}$ be the eigenspace of the operator in the bracket in (4.4) corresponding to negative eigenvalues. Then from the compactness assumption it follows that $\mathfrak{F}$ is finite dimensional. Since V is symmetric we see that the operator in the bracket is nonnegative on $\mathfrak{F}^{\perp}$, the orthocomplement of $\mathfrak{F}$. First we maintain that

$$(1 - \Delta + V) \geqslant 0 \qquad \text{on} \qquad (1 - \Delta)^{-1/2} \mathfrak{F}^{\perp}. \tag{4.5}$$

For, setting

$$g = (1 - \Delta)^{-1/2} f^{\perp}, \qquad f^{\perp} \in \mathfrak{F}^{\perp}$$

we see from equation (4.4) that

$$(g, (1 - \Delta - V)g) = (f^\perp, [1 + (1 - \Delta)^{-1/2} V(1 - \Delta)^{-1/2}] f^\perp).$$

Remembering the definition of $\mathfrak{F}^\perp$ we obtain the validity of relation (4.5). Next we maintain that

$$\dim\{((1 - \Delta)^{-1/2} \mathfrak{F}^\perp)^\perp\} < \infty. \tag{4.6}$$

For, suppose that the vector h is in this ortho-complement. That is, suppose that for every vector $f^\perp$ in $\mathfrak{F}^\perp$,

$$(h, (1 - \Delta)^{-1/2} f^\perp) = ((1 - \Delta)^{-1/2} h, f^\perp) = 0.$$

Then clearly, we have

$$(1 - \Delta)^{-1/2} h \in (\mathfrak{F}^\perp)^\perp = \mathfrak{F},$$

if we remember that $\mathfrak{F}$ is finite dimensional. Thus

$$(1 - \Delta)^{-1/2} \{(1 - \Delta)^{-1/2} \mathfrak{F}^\perp\}^\perp \subset \mathfrak{F},$$

and we obtain the validity of relation (4.6). Then combining relations (4.6) and (4.5) we arrive at the validity of Lemma 4.1.

Next recall relation (4.2) which allows us to apply Lemma 4.1 to the operator

$$V = M(p).$$

Then from this Lemma we conclude that the sesquilinear form of the operator $1 - \Delta + M(p)$ on $\ddot{\mathfrak{C}}_\infty$ is bounded from below. In other words the first conclusion of Theorem 2.1 is established.

5. Estimates for the Operator Valued Green's Function of the Laplacian

If we separate the angular and radial variables in the Green's function of the Laplacian we obtain an operator which acts on the angular variables only. We call this operator, the operator valued Green's function and estimates for its norm will play a key role in the proof of Theorem 2.1. To describe it more specifically we need some notations.

Let $\mathscr{S}_{d-1}$ denote the $(d-1)$-dimensional unit sphere and $\mathfrak{L}_2(\mathscr{S}_{d-1})$ the

$\mathfrak{L}_2$-space over it. We shall call $\mathfrak{L}_2(\mathscr{S}_{d-1})$ the accessory space and for an element φ in $\mathfrak{L}_2(\mathscr{S}_{d-1})$ we set

$$| \varphi |_{\mathfrak{A}} = \left(\int_{|u|=1} | \varphi(u) |^2 \, dS_u \right)^{1/2}. \tag{5.1}$$

For an operator t on $\mathfrak{L}_2(\mathscr{S}_{d-1})$ we denote by $| t |_{\mathfrak{A}}$ its norm, i.e. set

$$| t |_{\mathfrak{A}} = \sup_{\varphi} \frac{| t\varphi |_{\mathfrak{A}}}{| \varphi |_{\mathfrak{A}}}. \tag{5.2}$$

We also set

$$| (\varphi\psi)_{\mathfrak{A}} | = | \bar{\varphi}\psi |_{\mathfrak{A}}. \tag{5.3}$$

Let the transformation T mapping $\mathfrak{L}_2(\mathscr{E}_d)$ into $\mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathscr{S}_{d-1}))$ be defined by

$$Tf(\xi) = \xi^{(d-1)/2} f(\xi u), \qquad f \in \mathfrak{L}_2(\mathscr{E}_d), \quad u \in \mathscr{S}_{d-1}. \tag{5.4}$$

The definition of the induced spherical measure shows that T is an isometry. At the same time it follows that T is onto, i.e.

$$T\mathfrak{L}_2(\mathscr{E}_d) = \mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathscr{S}_{d-1})).$$

Clearly, the adjoint is given by

$$T^*f(x) = \left(\frac{1}{| x |} \right)^{(d-1)/2} f(| x |) \left(\frac{x}{| x |} \right), f \in \mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathscr{S}_{d-1}) \; x \in \mathscr{E}_d. \tag{5.4*}$$

As is well known the spectrum of $-\Delta$ consists of the positive axis [B.9]. At every nonreal complex number μ, the resolvent $(\mu + \Delta)^{-1}$ is an integral-operator. That is, there is a function $(\mu + \Delta)^{-1}(x, y)$ of the variables x, y in $\mathscr{E}_d \times \mathscr{E}_d$, such that for every f in $\mathfrak{L}_2(\mathscr{E}_d)$,

$$(\mu + \Delta)^{-1} f(x) = \int (\mu + \Delta)^{-1}(x, y) f(y) \, dy. \tag{5.5}$$

Definitions (5.4), (5.4*) and (5.5), together with the definition of the spherical measure show that

$$T(\mu + \Delta)^{-1} T^*f(\xi)(u) = \iint (\xi\eta)^{(d-1)/2} (\mu + \Delta)^{-1} (\xi u, \eta v) f(\eta)(v) \, dS_v d\eta,$$
$$f \in \mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathscr{S}_{d-1})), \; u \in \mathscr{S}_{d-1}.$$

For frozen ξ, η and μ consider the operator on $\mathfrak{L}_2(\mathscr{S}_{d-1})$ defined by the kernel

$$T(\mu+\Delta)^{-1}\,T^*(\xi,\eta)\,(u,v)=(\xi\eta)^{(d-1)/2}\,(\mu+\Delta)^{-1}\,(\xi u,\eta v) \qquad (5.6)$$
$$u,v\in\mathscr{S}_{d-1}\times\mathscr{S}_{d-1}\,.$$

Then according to the previous equation, for every f in $\mathfrak{L}_2((0,\infty),\mathfrak{L}_2(\mathscr{S}_{d-1}))$

$$T(\mu+\Delta)^{-1}\,T^*f(\xi)=\int T(\mu+\Delta)^{-1}\,T^*(\xi,\eta)\,f(\eta)\,d\eta. \qquad (5.7)$$

In other words the $\mathfrak{L}_2(\mathscr{S}_{d-1})$ operator valued function $T(\mu+\Delta)^{-1}\,T^*(\xi,\eta)$ is the kernel of the operator on the left. We call the function the Green's function of the Laplacian. Note that it depends on the three variables μ and (ξ,η). Actually for fixed (ξ,η) we have two functions of μ. One for μ in the upper half plane and one for μ in the lower half plane. As in Section 3 we denote the points of these half planes by $\mu_\pm$ and set

$$K^\pm(\mu_\pm)\,(\xi,\eta)=\begin{cases}T(\mu_+ +\Delta)^{-1}\,T^*(\xi,\eta) & \mu_+\in\mathscr{R}_+(\mathscr{I})\\ T(\mu_- +\Delta)^{-1}\,T^*(\xi,\eta) & \mu_-\in\mathscr{R}_-(\mathscr{I}).\end{cases} \qquad (5.8)$$

In the two theorems that follow we formulate two key properties of the operator valued Green's function.

THEOREM 5.1. *Let the operator $K^\pm(\mu_\pm)\,(\xi,\eta)$ on $\mathfrak{L}_2(\mathscr{S}_{d-1})$ be defined by relation* (5.8). *Suppose that $\mathscr{I}$ is a compact interval which does not contain zero. Then there is a constant $O(1)$, such that for every pair of positive numbers (ξ,η),*

$$\sup_{\mu_\pm\in\mathscr{R}_\pm(\mathscr{I})}|\,K^\pm(\mu_\pm)\,(\xi,\eta)\,|_{\mathfrak{A}}=O(1)\begin{Bmatrix}1, & d=1\\ \min(\xi^{1/2},\eta^{1/2},\xi^{1/3},\eta^{1/3}), & d=2\\ \min(\xi,\eta,\xi^{1/3},\eta^{1/3}), & d\geqslant 3\end{Bmatrix}. \qquad (5.9)$$

THEOREM 5.2. *The operators $K^\pm(\mu_\pm)\,(\xi,\eta)$ of relation* (5.9) *depend continuously in the $\mathfrak{L}_2(\mathscr{S}_{d-1})$ operator-norm on the three variables $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and any (ξ,η) in $[0,\infty)\times[0,\infty)$. Furthermore in the variables $\mu_\pm$ they can be continuously extended onto the closures $\mathscr{R}_\pm(\mathscr{I})$.*

We establish these two theorems together. The joint proof makes essential use of the fact that $T\Delta T^*$ admits a family of reducing subspaces and on each of them it acts like an ordinary differential operator. To describe this in more specific terms we need the notion of the Laplace-Beltrami operator acting in $\mathfrak{L}_2(\mathscr{S}_{d-1})$. It was emphasized by Kato [B.1] that this is the operator A

determined by the requirement that for every smooth function f in $\mathfrak{L}_2((0, \infty), \mathfrak{L}_2(\mathscr{S}_{d-1}))$,

$$T\Delta T^* f(\xi) = f''(\xi) - \frac{1}{\xi^2}\left[A + \frac{(d-1)(d-3)}{4}\right] f(\xi). \tag{5.10}$$

As is well known [B.13], A is self-adjoint, its spectrum is discrete, and it is given by

$$\lambda_l = \begin{cases} 0 & d = 1, \\ l(l + d - 2) & d \geqslant 2, \qquad l = 0, 1, 2, \dots . \end{cases} \tag{5.11}$$

Let $\mathfrak{E}_l$ denote the eigenspace of A corresponding to the eigenvalue λ_l. Let o_l denote the orthoprojector on $\mathfrak{E}_l$ i.e., set

$$o_l \mathfrak{L}_2(\mathscr{S}_{d-1}) = \mathfrak{E}_l , \qquad o_l = \bar{o}_l , \qquad o_l^2 = o_l$$

where $\bar{o}_l$ denotes the adjoint of o_l. Since the spectral projectors of a self-adjoint operator corresponding to different eigenvalues are orthogonal, the operator

$$j_n = \sum_{l=0}^{n-1} o_l , \qquad j_0 = 0 \tag{5.12$_n$}$$

is also a projector and clearly it is of finite rank.

It is clear from equation (5.10) that the subspace $\mathfrak{L}_2((0, \infty), \mathfrak{E}_l)$ reduces the operator $T\Delta T^*$. At the same time we see, that if the function f_l is of the form

$$f_l(x) = \varphi_l(|\, x\,|)\, e_l\left(\frac{x}{|\, x\,|}\right), \qquad e_l \in \mathfrak{E}_l ,$$

then

$$T\Delta T^* f_l(x) = \left[\varphi_l''(|\, x\,|) - \frac{1}{|\, x\,|^2}\left(\lambda_l + \frac{(d-1)(d-3)}{4}\right)\varphi(|\, x\,|)\right] e_l\left(\frac{x}{|\, x\,|}\right). \tag{5.13}$$

Next let L_l be the $\mathfrak{L}_2(0, \infty)$-closure of the operator defined by

$$L_l \varphi(\rho) = \varphi''(\rho) - \frac{1}{\rho^2}\left(\lambda_l + \frac{(d-1)(d-3)}{4}\right)\varphi(\rho), \tag{5.14}$$

for those complex valued smooth functions which satisfy the boundary condition $\varphi(0) = 0$ and vanish at infinity. For brevity we denote the set of such functions by $\mathfrak{D}_\infty$. In the lemma that follows, we formulate a connection between the Green's functions of the operators $\{L_l\}$, the projector j_n of definition (5.12) and the operator valued Green's function of Δ.

LEMMA 5.1. *Let the operator L_l be defined by equation* (5.14), *and for each nonreal complex number μ let $(\mu + L_l)^{-1}(\xi, \eta)$ denote the value of the kernel of $(\mu + L_l)^{-1}$ at the point (ξ, η). Then for the operator valued Green's function of Δ we have*

$$| (\mu + T\Delta T^*)^{-1}(\xi, \eta) - j_n(\mu + T\Delta T^*)^{-1}(\xi, \eta) |_{\mathfrak{U}} \leqslant \sup_{m \geqslant n} | (\mu + L_m)^{-1}(\xi, \eta) | . \tag{5.15}$$

To verify this lemma recall definition (5.14) which shows, according to general considerations [B.7.i], that the operator L_l on $\mathfrak{D}_\infty$ is essentially self-adjoint. This, in turn, shows that for each nonreal μ,

$$\overline{(\mu + L_l)\,\mathfrak{D}_\infty} = \mathfrak{L}_2(0,\infty),$$

where the bar denotes closure. For a given linear subset $\mathfrak{S}$ of $\mathfrak{L}_2(0, \infty)$ let $\mathfrak{S} \cdot \mathfrak{E}_l$ denote the set of finite linear combinations of the form

$$\sum \varphi_n e_n , \qquad \varphi_n \in \mathfrak{S}, \qquad e_n \in \mathfrak{E}_l .$$

Then we see from the previous relation that, for each l

$$\overline{(\mu + L_l)\,\mathfrak{D}_\infty \cdot o_l} = \mathfrak{L}_2((0, \infty), \mathfrak{E}_l). \tag{5.16}$$

We maintain that for each (ξ, η) and l

$$(\mu + L_l)^{-1}(\xi, \eta) \cdot o_l = T(\mu_\pm + \Delta)^{-1}\, T^*(\xi, \eta) \cdot o_l . \tag{5.17}$$

Note that the left member is the product of a complex number with o_l, while the right member is the product of an operator on $\mathfrak{L}_2(\mathscr{S}_{d-1})$ with o_l. To verify relation (5.17), we first claim that for every function f of the form

$$f(\eta) = (\mu + L_l)\,\varphi(\eta)\, e, \qquad \varphi \in \mathfrak{D}_\infty , \qquad e \in \mathfrak{E}_l , \tag{5.18}$$

we have

$$\int (\mu + L_l)^{-1}(\xi, \eta) \cdot f(\eta)\, d\eta = \int T(\mu + \Delta)^{-1}\, T^*(\xi, \eta) f(\eta)\, d\eta. \tag{5.19}$$

It is clear from definition (5.18) that

$$\int (\mu + L_l)^{-1}(\xi, \eta) f(\eta)\, d\eta = \varphi(\xi) \cdot e. \tag{5.19}$$

To evaluate the right member of (5.19) recall equation (5.13) and definition (5.14). Then remembering that T is unitary we see that

$$T(\mu + \Delta)^{-1}\, T^*(\varphi(\eta) \cdot e) = (\mu + L_l)^{-1}\, \varphi(\eta) \cdot e.$$

This equation together with definition (5.18) inserted in the right member of (5.19) yields

$$\int T(\mu + \Delta)^{-1} T^*(\xi, \eta) f(\eta) \, d\eta = \varphi(\xi) \cdot e. \tag{5.19$_r$}$$

Combining relations $(5.19)_l$ and $(5.19)_r$ we obtain the validity of relation (5.19). According to relation (5.16) vectors of the form (5.18) are dense in $\mathfrak{L}_2((0, \infty), \mathfrak{E}_l)$. Since the operators in (5.19) are bounded this equation holds on all of $\mathfrak{L}_2((0, \infty), \mathfrak{E}_l)$. Since the kernel of an integral operator is uniquely determined by the operator, we arrive at the validity of relation (5.17).

Relation (5.17) and definition (5.12) together show that for each n and m

$$\begin{aligned} &| j_n T(\mu + \Delta)^{-1} T^*(\xi, \eta) - j_m T(\mu + \Delta)^{-1} T^*(\xi, \eta) |_{\mathfrak{A}} \\ &\qquad \leqslant \sup_{n<k<m} | (\mu + L_k)^{-1} (\xi, \eta) | \end{aligned}$$

if we remember the orthogonality of the spectral projectors. Finally taking the limit as m tends to infinity we arrive at the validity of conclusion (5.15). This completes the proof of Lemma 5.1.

Next we describe the kernels in Lemma 5.1. For each l and d, define the positive number ν by

$$\lambda_l + \tfrac{1}{4}(d-1)(d-3) = \nu^2 - \tfrac{1}{4}$$

Insertion of relation (5.11) in this equation yields

$$\begin{aligned} &\nu^2 = \tfrac{1}{4}, && d = 1 \\ &\nu^2 = l(l + d - 2) + \frac{(d-2)^2}{4} && d \geqslant 2 \qquad l = 0, 1, 2, \ldots . \end{aligned} \tag{5.20}$$

Clearly as l ranges over the nonnegative integers ν ranges over a discrete set of positive values, whose minimum is attained for $l = 0$. Hence

$$\nu \geqslant \frac{d-2}{2}, \qquad d \geqslant 2. \tag{5.21}$$

Insertion of equation $(5.18)_\nu$ in equation (5.13) yields

$$L_l \varphi(\xi) = -\varphi''(\xi) + \frac{1}{\xi^2}\left(\nu^2 - \frac{1}{4}\right)\varphi(\xi). \tag{5.22$_l$}$$

We maintain that the kernel of this sequence of operators is given for μ_+ in $\mathscr{R}_+(\mathscr{I})$ by

$$(\mu_+ + L_l)^{-1}(\xi, \eta) = \frac{\pi}{2i} \begin{cases} \sqrt{\xi}\, H_\nu^1(\xi\sqrt{\mu_+})\, \sqrt{\eta}\, J_\nu(\eta\sqrt{\mu_+}), & \eta < \xi \\ \sqrt{\xi}\, J_\nu(\xi\sqrt{\mu_+})\, H_\nu^1(\eta\sqrt{\mu_+}), & \eta > \xi \end{cases} \tag{5.23$_+$}$$

and for μ_- in $\mathscr{R}_-(\mathscr{I})$ by

$$(\mu_- + L_l)^{-1}(\xi, \eta) = \frac{\pi}{2i}\begin{cases}\sqrt{\xi}\, H_\nu^2(-\,\xi\sqrt{\mu_-})\sqrt{\eta}\, J_\nu(-\,\eta\sqrt{\mu_-}), & \eta < \xi\\ \sqrt{\xi}\, J_\nu(-\,\xi\sqrt{\mu_-})\sqrt{\eta}\, H_\nu^2(-\,\eta\sqrt{\mu_-}), & \eta > \xi.\end{cases} \tag{5.23$_-$}$$

Here $\sqrt{\mu}$ denotes that branch of this multi-valued function for which

$$\operatorname{Im}\sqrt{\mu} \geqslant 0 \qquad \mu \notin [0, \infty). \tag{5.24}$$

The functions J_ν and $H_\nu^{1,2}$ are Bessel functions in the usual notations [B.8]. More specifically the functions

$$\sqrt{z}\, J_\nu(z), \qquad \sqrt{z}\, H_\nu^1(z), \qquad \sqrt{z}\, H_\nu^2(z)$$

satisfy the differential equation, [B.12.a],

$$y''(z) - \frac{1}{z^2}\left(\nu^2 - \frac{1}{4}\right)y(z) + y(z) = 0. \tag{5.25}$$

The adjusted Hankel functions $\sqrt{z}\, H_\nu^{1,2}(z)$ are those solutions whose asymptotic descriptions in the plane cut along the negative axis are given by

$$\sqrt{z}\, H_\nu^{(1,2)}(z) \sim \sqrt{\frac{2}{\pi}}\exp\left(\mp\, i\pi\,\frac{2\nu+1}{4}\right)\exp(\pm\, iz)$$

$$|\, z\, | \sim \infty \qquad z \notin (-\,\infty, 0]. \tag{5.26^{1,2}}$$

The existence of such solutions to equation (5.25) is ensured by the theory of the irregular singular point [B.8]. The adjusted Bessel function $\sqrt{z}\; J_\nu(z)$ is the solution of equation (5.25) whose asymptotic expansion is given near the point $z = 0$. This expansion involves a constant that depends on ν, and we shall not specify it or the expansion. We shall need two well known consequences of this expansion. The first is that for each positive number ν the function $\sqrt{z}\; J_\nu(z)$ is bounded near $z = 0$. The second consequence is rather delicate and it says that $\sqrt{z}\; J_\nu(z)$ is a linear combination of the adjusted Hankel functions with ν-independent coefficients. Specifically we have [B.12.b],

$$\sqrt{z}\; J_\nu(z) = \tfrac{1}{2}\sqrt{z}\, H_\nu^1(z) + \tfrac{1}{2}\sqrt{z}\, H_\nu^2(z). \tag{5.27}$$

This is the so called connection formula for the Bessel function J_ν. Note that in general the symbols J_ν and $H_\nu^{1,2}$ denote multi-valued analytic functions and we use them to denote particular single-valued branches.

As is well known [B.12.a] these definitions imply that the functions

$$\sqrt{\rho}\, J_\nu(\rho \sqrt{\mu}), \qquad \sqrt{\rho}\, H_\nu^1(\rho \sqrt{\mu}), \qquad \sqrt{\rho}\, H_\nu^2(\rho \sqrt{\mu}),$$

satisfy the differential equation

$$\mu g(\rho) + \frac{d^2}{d\rho^2} g(\rho) - \frac{\nu^2 - \frac{1}{4}}{\rho^2} g(\rho) = 0. \tag{5.28}$$

We have seen that the functions $J_\nu(\rho \sqrt{\mu_\pm})$ are bounded and hence square integrable near $\rho = 0$. Similarly we see from formulae (5.24) and $(5.26)^{1,2}$ that for μ_+ in $\mathscr{R}_+(\mathscr{I})$ the function $\rho H_\nu^1(\rho \sqrt{\mu_+})$ is square integrable near $\rho = \infty$, and for μ_- in $\mathscr{R}_-(\mathscr{I})$ so is $\rho H_\nu^2(-\rho \sqrt{\mu_-})$. These facts together with the Weyl representation theorem for the resolvent of an ordinary differential operator [B.7.f] yield the validity of formulae $(5.23)_\pm$ up to a multiplicative constant. Since we shall make essential use of the fact that this constant is independent of ν we evaluate it. According to the representation theorem [B.7.f], to complete the proof of these formulae we have to show that for each complex number μ the Wronskian of these functions equals the constant in $(5.23)_\pm$. That is

$$\begin{vmatrix} \sqrt{\rho}\, J_\nu(\pm \rho \sqrt{\mu_\pm}) & \sqrt{\rho}\, H_\nu^{1,2}(\pm \rho \sqrt{\mu_\pm}) \\ \frac{d}{d\rho} \sqrt{\rho}\, J_\nu(\pm \rho \sqrt{\mu_\pm}) & \frac{d}{d\rho} \sqrt{\rho}\, H_\nu^{1,2}(\pm \rho \sqrt{\mu_\pm}) \end{vmatrix} = \frac{2i}{\pi}. \tag{$5.29_\pm$}$$

For, from the connection formula (5.27) we see that for each complex number z in the cut plane

$$\begin{vmatrix} \sqrt{z}\, J_\nu(z) & \sqrt{z}\, H_\nu^{1,2}(z) \\ \frac{d}{dz} \sqrt{z}\, J_\nu(z) & \frac{d}{dz} \sqrt{z}\, H_\nu^{1,2}(z) \end{vmatrix} = \frac{1}{2} \begin{vmatrix} \sqrt{z}\, H_\nu^{2,1}(z) & \sqrt{z}\, H_\nu^{1,2}(z) \\ \frac{d}{dz} \sqrt{z}\, H_\nu^{2,1}(z) & \frac{d}{dz} \sqrt{z}\, H_\nu^{1,2}(z) \end{vmatrix}. \tag{5.30}$$

This determinant is a Wronskian of equation (5.25) which does not contain a first order term. Hence it does not depend on z and it suffices to evaluate it near $z = \infty$. Since this point is an irregular singular point [B.8] of equation (5.25) we can differentiate the asymptotic formulae $(5.26)^{1,2}$. This yields

$$\frac{d}{dz} \sqrt{z}\, H_\nu^{1,2}(z) \sim \pm i \sqrt{\frac{2}{\pi}} \exp\left[\pm i \left(z - \frac{2\nu + 1}{4} \pi\right)\right]$$

$$|z| \sim \infty, \qquad z \in (-\infty, 0].$$

Insertion of these formulae and of $(5.26)^{1,2}$ in (5.30) yields, for each z in the cut plane,

$$\begin{vmatrix} \sqrt{z}\, J_\nu(z) & \sqrt{z}\, H_\nu^{1,2}(z) \\ \frac{d}{dz}\sqrt{z}\, J_\nu(z) & \frac{d}{dz}\sqrt{z}\, H_\nu^{1,2}(z) \end{vmatrix} = \pm \frac{2i}{\pi}. \tag{5.31}$$

Since ρ is positive, we see from definition (5.24) that for $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$,

$$\operatorname{Re}\,(\pm\,\rho\sqrt{\mu_\pm}) > 0.$$

Hence we can set $z = \pm\,\rho\sqrt{\mu_\pm}$ in relation (5.31). Remembering the chain rule this establishes the validity of relations $(5.29)_\pm$. This in turn, establishes the validity of relations $(5.23)_\pm$.

Remembering definition (5.8) and inserting relations $(5.23)_\pm$ in Lemma 5.1, for each n we obtain the two key relations,

$$| K^+(\mu_\pm)\,\xi, \eta) - j_n K^+(\mu_\pm)\,(\xi, \eta)\,|_{\mathfrak{A}}$$

$$\leqslant \frac{\pi}{2}\begin{cases} \sup\,|\,(\xi\eta)^{1/2} H_\nu^{1,2}(\pm\,\xi\sqrt{\mu_\pm})\, J_\nu(\pm\,\eta\sqrt{\mu_\pm})\,|, & \eta \leqslant \xi \\ \sup\,|\,(\xi\eta)^{1/2} J_\nu(\pm\,\xi\sqrt{\mu_\pm})\, H_\nu^{1,2}(\pm\,\eta\sqrt{\mu_\pm})\,|, & \eta \geqslant \xi. \end{cases} \tag{$5.32)_{n,\pm}$$

Here the supremum is taken for those values of ν which satisfy equation (5.20) and are greater than n.

To derive conclusion (5.9) from relations $(5.32)_{n,\pm}$ we need uniform estimates for the Bessel functions on the right. The question of such estimates, for complex order and complex arguments, was investigated by Olver [B.10]. All that we need however, are two corollaries of his deep results concerning the special case of positive order and arguments in an appropriate guadrant of the right-halfplane. These two corollaries were formulated and proved elsewhere [B.11]. According to the first of these corollaries [B.11] there is a constant $O(1)$ such that for every (ξ, η) in $[0, \infty) \times [0, \infty)$ and $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ the right members of $(5.32)_{n,\pm}$ are majorized by

$$O(1)\begin{Bmatrix} 1, & d = 1 \\ \min(\xi^{1/2}, \eta^{1/2}, \xi^{1/3}, \eta^{1/3}), & d = 2 \\ \min(\xi, \eta, \xi^{1/3}, \eta^{1/3}), & d \geqslant 3 \end{Bmatrix}.$$

Thus we arrive at the validity of conclusion (5.9), if we remember definition $(5.12)_0$ which says that $j_0 = 0$. This completes the proof of Theorem 5.1.

To verify Theorem 5.2 recall definitions (5.8), $(5.12)_n$ and relation (5.17). They show that

$$j_n K^{\pm}(\mu_{\pm})(\xi, \eta) = \sum_{l=0}^{n-1} (\mu_{\pm} + L_l)^{-1} (\xi, \eta) \cdot o_l . \tag{5.33}$$

According to formulae $(5.23)_{\pm}$ the first factors on the right depend continuously on $\mu_{\pm}$ in $\mathscr{R}_{\pm}(\mathscr{I})$ and (ξ, η) in $[0, \infty) \times [0, \infty)$. At the same time we see that in the variables $\mu_{\pm}$ they can be continuously extended onto the closures $\overline{\mathscr{R}_{\pm}(\mathscr{I})}$. Since $\{o_l\}$ is a family of mutually orthogonal ortho-projectors on $\mathfrak{L}_2(\mathscr{S}_{d-1})$, formula (5.33) shows that the left member has these continuity properties. According to the second corollary formulated elsewhere [B.11], the right member of $(5.32)_{n,\pm}$ tends to zero as n tends to infinity. This convergence is uniform in $\mu_{\pm}$ in the closures $\mathscr{R}_{\pm}(\mathscr{I})$ and (ξ, η) in any compact subset. Thus according to relations $(5.32)_{n,\pm}$, the kernels $K^{\pm}(\mu_{\pm})(\xi, \eta)$ can be represented as limits of continuous kernels. Since this limit is uniform with reference to the $\mathfrak{L}_2(\mathscr{S}_{d-1})$-operator norm, this kernel is continuous in the sense of Theorem 5.2. This completes the proof of Theorem 5.2.

6. Proof of Theorem 2.1

In this section we derive Theorem 2.1 from the abstract Theorem 3.1. For the unperturbed operator we take $-\Delta$ and for the perturbed operator we take $(-\Delta + M(p))_F$. Then with the aid of the potential p we construct a gentleness norm, with reference to which the assumptions of Theorem 3.1 hold for this pair of operators.

Our gentleness norm will consist of three terms. We define $\|f\|_1$ to be the smallest number such that for every x in $\mathscr{E}_d$

$$|f(x)| \leqslant \|f\|_1 \, |p_1(x)| . \tag{6.1$_1$}$$

In order to define $\|f\|_2$ we need an elementary fact which is the consequence of an observation made elsewhere [A.14.a]. Specifically we need that Condition II implies the existence of a positive function w_d, such that it is bounded away from zero on any compact set which does not contain zero,

$$\lim_{\xi=0} \frac{1}{w_{d(\xi)}} \begin{Bmatrix} 1, & d=1 \\ \xi^{1/2}, & d=2 \\ \xi, & d \geqslant 3 \end{Bmatrix} = \lim_{\xi=\infty} \frac{1}{w_{d(\xi)}} = 0, \tag{6.2$_{0,\infty}$}$$

and

$$\int_0^1 p_2^*(\xi)\, w_d(\xi)\, d\xi + \int_1^{\infty} p_2^*(\xi) \begin{Bmatrix} 1, & d=1 \\ |\xi^{1/3}|, & d \geqslant 2 \end{Bmatrix} w_d^2(\xi)\, d\xi < \infty. \tag{6.3}$$

At first the integrand in (6.3) may appear as w_d-times the integrand in Condition II; this, however, is not the case. With the aid of such a function w_d and of definitions (2.1), (5.1) and (5.5) we define $\|f\|_2$ to be the smallest number such that for every ξ

$$| Tf(\xi) |_{\mathfrak{A}} \leqslant \|f\|_2 \cdot p_2^*(\xi)\, w_d(\xi). \tag{6.1$_2$}$$

Next with the aid of relation (2.2) we set

$$\|f\|_3 = I_{d,\delta}(f), \tag{6.1$_3$}$$

where δ is such that the support of p_1 is contained in the ball $| x | \leqslant \delta$, denoted by $\mathscr{B}_\delta$. Since according to the Appendix, Condition II implies Condition I, it is no loss of generality to assume that the support of p_2 is disjoint from this ball. Let $P_{1,2}$ be the projectors on the space of measurable functions which project onto this ball and its complement. That is, set

$$P_1 f(x) = \begin{cases} f(x), & | x | < \delta \\ 0, & | x | > \delta \end{cases} \qquad P_2 f(x) = \begin{cases} 0, & | x | < \delta \\ f(x), & | x | > \delta \end{cases}. \tag{6.4$_{1,2}$}$$

Finally, with the aid of relations $(6.1)_{1,2,3}$ and $(6.4)_{1,2}$ we define

$$\|f\|_{\mathfrak{B}} = \| P_1 f\|_1 + \| P_2 f\|_2 + \|f\|_3 , \tag{6.1}$$

and denote by $\mathfrak{B}$ the space of those measurable functions for which this norm is finite. Note that if p_1 is indentically zero then the first and third terms in (6.1) are also zero.

To verify that the assumptions of Theorem 3.1 hold for the pair $(-\Delta, (-\Delta + M(p))_F)$ with reference to this norm, we need estimates for the Green's function of the Laplacian. Recall definition (5.4) and set

$$(\mu_\pm + \Delta)^{-1} (x, y) = R_0^\pm(\mu_\pm) (x, y). \tag{6.5}$$

It is well known [B.6] that the Green's function can be written as

$$R_0^\pm(\mu_\pm) (x, y) = k^\pm(\mu_\pm , | x - y |), \tag{6.6}$$

where $k^\pm$ are such that for any compact interval $\mathscr{I}$, which does not contain zero, we have the three relations that follow.

For each x, y with $x \neq y$, the two functions $k^\pm(\mu_\pm , | x - y |)$ of the variables $\mu_\pm$ are continuous in the open sets $\mathscr{R}_\pm(\mathscr{I})$) and can be continuously extended onto the closures $\overline{\mathscr{R}_\pm(\mathscr{I})}$. The extended functions are continuous on any compact subset of

$\mathscr{E}_d \times \mathscr{E}_d$ which does not contain the diagonal $x = y$ and $\omega_\pm$ in the closure $\overline{\mathscr{R}_\pm(\mathscr{I})}$. (6.7)

To each (x, y) region in which $|x - y|$ is bounded from above there is a constant $O(1)$ such that for every $\omega_\pm$ in $\overline{\mathscr{R}_\pm(\mathscr{I})}$

$$k^\pm(\omega_\pm, |x-y|) = O(1) \begin{Bmatrix} 1, & d = 1 \\ \log \dfrac{1}{|x-y|}, & d = 2 \\ \left(\dfrac{1}{|x-y|}\right)^{d-2}, & d \geqslant 3 \end{Bmatrix}. \tag{6.8$_0$}$$

To each (x, y) region in which $|x - y|$ is bounded from below there is a constant $O(1)$ such that for every $\omega_\pm$ in $\overline{\mathscr{R}_\pm(\mathscr{I})}$

$$k^\pm(\omega_\pm, |x-y|) = O(1) \left(\frac{1}{|x-y|}\right)^{(d-1)/2}. \tag{6.8$_\infty$}$$

After these preparations we verify the conditions of Theorem 3.1.

a. Condition $G_1(\mathscr{I})$. Relation (6.7) allows us to define for each $\omega_\pm$ in the closure $\overline{\mathscr{R}_\pm(\mathscr{I})}$ the extended Green's function $R_0^\pm(\omega_\pm)(x, y)$. The lemma that follows will imply that these kernels define bounded forms on $\mathfrak{B} \times \mathfrak{B}$ for which Condition $G_1(\mathscr{I})$ holds. We denote these forms by $[R_0^\pm(\omega_\pm)]_{\mathfrak{B}}$. Note that Condition $G_1(\mathscr{I})$ involves only the unperturbed operator and it holds with reference norms which are more general then (6.1). But we shall not be concerned with this fact.

Lemma 6.1. *Let the space $\mathfrak{B}$ be defined by relation* (6.1) *and let $\mathscr{I}$ be a compact interval which does not contain zero. Suppose that the potential p_1 has bounded support, it satisfies Condition* I, *and that p_2 satisfies Condition* II. *Then there is a constant $O(1)$ such that for every $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and f, g in $\mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d) \times \mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d)$*

$$|(f, R_0^\pm(\mu_\pm) g)| = O(1) \|f\|_{\mathfrak{B}} \cdot \|g\|_{\mathfrak{B}}, \tag{6.9}$$

and for each $\omega_\pm$ in the closure $\overline{\mathscr{R}_\pm(\mathscr{I})}$,

$$\lim_{\mu_\pm \to \omega_\pm} [R_0^\pm(\mu_\pm)]_{\mathfrak{B}}(f, g) = [R_0^\pm(\omega_\pm)]_{\mathfrak{B}}(f, g), \tag{6.10}$$

uniformly in $\omega_\pm$.

It is clear from definitions $(6.4)_{1,2}$ that

$$(f, R_0^{\pm}(\mu_{\pm})\, g) = (P_1 f, R_0^{\pm}(\mu_{\pm})\, P_1 g) + (P_2 f, R_0^{\pm}(\mu_{\pm})\, P_2 g) + (P_2 f, R_0^{\pm}(\mu_{\pm})\, P_1 g)$$
$$+ (P_1 f, R_0^{\pm}(\mu_{\pm})\, P_2 g). \tag{6.11}$$

This formula shows that in order to establish conclusions (6.9) and (6.10) it suffices to establish them for each of these terms. This is carried out in the three propositions that follow.

PROPOSITION $(6.1)_{11}$. *There is a constant $O(1)$ such that for every $\mu_{\pm}$ in $\mathscr{R}_{\pm}(\mathscr{I})$ and f, g in $\mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d) \times \mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d)$*

$$| (P_1 f, R_0^{\pm}(\mu_{\pm})\, P_1 g) | = O(1)\, \| P_1 f \|_1 \cdot \| P_1 g \|_1 \qquad (6.9)_{11}$$

and for $\omega_{\pm}$ in the closures $\overline{\mathscr{R}_{\pm}(\mathscr{I})}$

$$\lim_{\mu_{\pm} = \omega_{\pm}} [R_0^{\pm}(\mu_{\pm})]_{\mathfrak{B}}\, (P_1 f, P_1 g) = [R_0^{\pm}(\omega_{\pm})]_{\mathfrak{B}}\, (P_1 f, P_1 g) \qquad (6.10)_{11}$$

uniformly in $\omega_{\pm}$.

To verify conclusion $(6.9)_{11}$ recall definition $(6.1)_1$ which shows that

$$| (P_1 f, R_0^{\pm}(\mu_{\pm})\, P_1 g) | \leqslant \| P_1 f \|_1 \cdot \| P_1 g \|_1 \iint | R_0^{\pm}(\mu_{\pm})\,(x, y)\, p_1(x)\, p_1(y) |\, dx\, dy. \qquad (6.12)_{11}$$

We see from definition (2.2), estimate $(6.8)_0$, and from the fact that p_1 has bounded support that

$$\sup_x \left\{ (1 + | x |^{(d-1)/2}) \cdot \int_{|x-y| < 2\delta} | R_0^{\pm}(\mu_{\pm})\,(x, y)\, p_1(y) |\, dy \right\} < \infty.$$

Here the supremum is taken in x over all of $\mathscr{E}_d$ although at present we need it only for $| x | < \delta$. Similarly, we see from estimate $(6.8)_{\infty}$ that

$$\sup_x \left\{ (1 + | x |^{(d-1)/2}) \int_{|x-y| > 2\delta} | R_0^{\pm}(\mu_{\pm})\,(x, y)\, p_1(y) |\, dy \right\} < \infty.$$

Combining these two estimates we obtain

$$\sup_x \left\{ (1 + | x |^{(d-1)/2}) \int | R_0^{\pm}(\mu_{\pm})\,(x, y)\, p_1(y) |\, dy \right\} < \infty. \tag{6.13}$$

As a first consequence of this estimate and of the fact that p_1 is integrable, we see that

$$\iint | R_0^{\pm}(\mu_{\pm})(x, y) p_1(x) p_1(y) | \, dx \, dy < \infty \tag{6.14}$$

and this holds uniformly in $\mu_{\pm}$ in $\mathscr{R}_{\pm}(\mathscr{I})$ since all the other estimates did. Insertion of this fact in inequality (6.12) yields the validity of conclusion $(6.9)_{11}$.

To verify conclusion $(6.10)_{11}$ recall that we defined $[R_0^{\pm}(\omega_{\pm})]_{\mathfrak{B}}$ by the extended kernel $R_0^{\pm}(\omega_{\pm})(x, y)$. Hence

$$| [R_0^{\pm}(\mu_{\pm})]_{\mathfrak{B}} (P_1 f, P_1 g) - [R_0^{\pm}(\omega_{\pm})]_{\mathfrak{B}} (P_1 f, P_1 g) |$$

$$\leqslant \| P_1 f \|_{\mathfrak{B}} \| P_1 g \|_{\mathfrak{B}} \iint | R_0^{\pm}(\mu_{\pm}(x, y) - R_0^{\pm}(\omega_{\pm})(x, y) | \cdot | p_1(x) p_1(y) | \, dx \, dy. \tag{6.15$_{11}$}$$

The arguments used to establish estimate (6.14) show that the integrand on the right admits an integrable majorant which does not depend on $\mu_{\pm}$ or $\omega_{\pm}$.

According to relation (6.7) the first factor converges to zero as $\mu_{\pm}$ converges to $\omega_{\pm}$ and this is uniform in $\omega_{\pm}$ in $\overline{\mathscr{R}_{\pm}(\mathscr{I})}$ and (x, y) in any compact subset of $\mathscr{E}_d \times \mathscr{E}_d$ which is disjoint from the diagonal $x = y$. The second factor of the integrand in $(6.15)_{11}$ is clearly integrable. These facts together show that the integral in $(6.15)_{11}$ converges to zero as $\mu_{\pm}$ converges to $\omega_{\pm}$ and that this is uniform in $\overline{\mathscr{R}_{\pm}(\mathscr{I})}$. This establishes conclusion $(6.10)_{11}$ and completes the proof of Proposition $(6.1)_{11}$.

Proposition $(6.1)_{22}$. *There is a constant $O(1)$ such that for every $\mu_{\pm}$ in $\mathscr{R}_{\pm}(\mathscr{I})$ and every f, g in $\mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d) \times \mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d)$,*

$$(P_2 f, R_0^{\pm}(\mu_{\pm}) P_2 g) = O(1) \| P_2 f \|_{\mathfrak{B}} \| P_2 g \|_{\mathfrak{B}}, \tag{6.9$_{22}$}$$

and for $\omega_{\pm}$ in the closures $\overline{\mathscr{R}_{\pm}(\mathscr{I})}$,

$$\lim_{\mu_{\pm} = \omega_{\pm}} [R_0^{\pm}(\mu_{\pm})]_{\mathfrak{B}} (P_2 f, P_2 g) = [R_0^{\pm}(\omega_{\pm})]_{\mathfrak{B}} (P_2 f, P_2 g) \tag{6.10$_{22}$}$$

uniformly in $\omega_{\pm}$.

This proposition is an elementary consequence of Theorems 5.1 and 5.2 as we shall show presently. According to definition (5.5) T is a unitary transformation. Hence

$$(P_2 f, R_0^{\pm}(\mu_{\pm}) P_2 g) = (T P_2 f, T R_0^{\pm}(\mu_{\pm}) T^* T P_2 g).$$

Remembering relation (5.7) this yields

$$|(P_2 f, R_0^{\pm}(\mu_{\pm}) P_2 g)| \leqslant \iint |\overline{TR_0^{\pm}(\mu_{\pm}) T^*(\xi, \eta) TP_2 f(\xi)} \cdot TP_2 g(\eta)|_{\mathfrak{A}} \, d\xi \, d\eta. \tag{6.12$_{22}$}$$

Let the function m_d be defined by

$$m_d(\xi) = \begin{cases} 1 & 0 < \xi \leqslant 1 \\ \xi^{1/6} & 1 \leqslant \xi \end{cases} \qquad \text{for} \qquad d \geqslant 2 \tag{6.16$_d$}$$

and

$$m_d(\xi) = 1 \qquad \text{for} \qquad d = 1. \tag{6.16$_1$}$$

Then Theorem 5.1 implies that for every (ξ, η)

$$|TR_0^{\pm}(\mu_{\pm}) T^*(\xi, \eta)|_{\mathfrak{A}} = O(1)\, m_d(\xi)\, m_d(\eta).$$

This estimate together with definition (6.1) shows that

$$|\overline{TR_0^{\pm}(\mu_{\pm}) T^*(\xi, \eta) \cdot TP_2 f(\xi)}\, TP_2 g(\eta)|_{\mathfrak{A}}$$

$$= O(1) \| P_2 f \|_2 \| P_2 g \|_2 \cdot m_d(\eta)\, p_2^*(\xi)\, p_2^*(\eta)\, w_d(\xi)\, w_d(\eta),$$

if we apply the Schwarz inequality in the accessory space $\mathfrak{L}_2(\mathscr{S}_{d-1})$. Since the second factor is the product of the values of the function $m_d p_2^* w_d$ at the points ξ and η, we obtain

$$\iint |\overline{TR_0^{\pm}(\mu_{\pm}) T^*(\xi, \eta) TP_2 f(\xi)} \cdot TP_2 g(\eta)|_{\mathfrak{A}} \, d\xi \, d\eta$$

$$= O(1) \| P_2 f \|_2 \cdot \| P_2 g \|_2 \left(\int_0^{\infty} m_d(\xi)\, p_2^*(\xi)\, w_d(\xi)\, d\xi \right)^2. \tag{6.17}$$

Relations (6.3) and (6.16) together show that the integral on the right is finite. Insertion of this fact and of estimate (6.17) in inequality $(6.12)_{22}$ yields the validity of conclusion $(6.9)_{22}$.

To verify conclusion $(6.10)_{22}$ let $\omega_{\pm}$ be in $\overline{\mathscr{R}_{\pm}(\mathscr{S})}$. An adaptation of the arguments leading to estimate (6.17) shows that

$$|[R_0^{\pm}(\mu_{\pm})]_{\mathfrak{B}} (P_2 f, P_2 g) - [R_0^{\pm}(\omega_{\pm})]_{\mathfrak{B}} (P_2 f, P_2 g)|$$

$$= O(1) \cdot \| P_2 f \|_2 \cdot \| P_2 g \|_2 \cdot \iint |TR_0^{\pm}(\mu_{\pm}) T^*(\xi, \eta) - TR_0^{\pm}(\omega_{\pm}) T^*(\xi, \eta)|_{\mathfrak{A}}$$

$$\cdot\, p_2^*(\xi)\, w_d(\xi)\, p_2^*(\eta)\, w_d(\eta)\, d\xi\, d\eta. \tag{6.15$_{22}$}$$

At the same time we see that the integrand admits an integrable majorant which is independent of $\mu_\pm$ and $\omega_\pm$. According to Theorem 5.2, as $\mu_\pm$ converges to $\omega_\pm$, the first factor of the integrand converges to zero, and this is uniform in $\omega_\pm$ in $\overline{\mathscr{R}_\pm(\mathscr{I})}$ and ξ, η in any compact set. The second factor is independent of $\mu_\pm$ and $\omega_\pm$ and according to relation (6.3) it is integrable. Insertion of these facts in $(6.15)_{22}$ yields the validity of conclusion $(6.10)_{22}$. This completes the proof of Proposition $(6.1)_{22}$.

PROPOSITION $(6.1)_{12}$. *There is a constant $O(1)$ such that for every $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and every f, g in $\mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d) \times \mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d)$,*

$$(P_1 f, R_0^\pm(\mu_\pm) P_2 g) = O(1) \| P_1 f \|_{\mathfrak{B}} \| P_2 g \|_{\mathfrak{B}}, \tag{$6.9)_{12}$$

and for $\omega_\pm$ in the closure $\overline{\mathscr{R}_\pm(\mathscr{I})}$,

$$\lim_{\mu_\pm = \omega_\pm} [R_0^\pm(\mu_\pm)]_{\mathfrak{B}} (P_1 f, P_2 g) = [R_0^\pm(\omega_\pm)]_{\mathfrak{B}} (P_1 f, P_2 g) \tag{$6.10)_{12}$$

uniformly in $\omega_\pm$.

To verify conclusion $(6.9)_{12}$ recall definition $(6.1)_1$, which shows that

$$| (P_1 f, R_0^\pm(\mu_\pm) P_2 g) | \leqslant \| P_1 f \|_{\mathfrak{B}} \cdot \iint | R_0^\pm(\mu_\pm)(x, y) p_1(x) P_2 g(y) | \, dx \, dy. \tag{$6.12)_{12}$$

Estimate $(6.8)_0$ together with definition (2.2) yields

$$\sup_x \left\{ \int_{|x-y|<2\delta} | R_0^\pm(\mu_\pm)(x, y) P_2 g(y) | \, dy \right\} = O(1) \| P_2 g \|_3. \tag{6.18}$$

We claim that there is a constant $O(1)$, such that

$$\sup_x \left\{ \int_{2|x-y|>|x|+\delta} | R_0^\pm(\mu_\pm)(x, y) P_2 g(y) | \, dy \right\} = O(1) \| P_2 g \|_2. \tag{6.19}$$

Here, as before, the supremum in x is taken over all of $\mathscr{E}_d$. For, the triangle inequality shows that,

$$n | x | + \delta < 2 | x - y | < (n+1) | x | + \delta \textit{ implies}$$

$$\frac{| y |}{| x - y |} \leqslant \frac{(n+3) | x | + \delta}{n | x | + \delta} < 4, \qquad \text{for} \qquad n = 1, 2, \ldots .$$

Hence

$$\int_{2|x-y|>|x|+\delta} \left(\frac{1}{|x-y|}\right)^{(d-1)/2} | P_2 g(y) | \, dy$$

$$\leqslant 4^{(d-1)/2} \sum_{n=1}^{\infty} \int_{n|x|+\delta<2|x-y|<(n+1)|x|+\delta} \left(\frac{1}{|y|}\right)^{(d-1)/2} | P_2 g(y) | \, dy.$$

Insertion of estimate $(6.8)_\infty$ in this inequality yields

$$\int_{2|x-y|>|x|+\delta} | R_0^{\pm}(\mu_\pm)(x, y) P_2 g(y) | \, dy = O(1) \int \left(\frac{1}{|y|}\right)^{(d-1)/2} | P_2 g(y) | \, dy.$$

Definitions (5.1), $(6.1)_2$ and (5.5) show that

$$\eta^{(d-1)/2} \int_{|v|=1} | P_2 g(\eta v) | \, dS_v = O(1) \, \| P_2 g \|_2 \, p_2^*(\eta) \, w_d(\eta),$$

if we apply the Schwarz inequality in the accessory space $\mathfrak{L}_2(\mathscr{S}_{d-1})$. These two estimates together with relation (6.3) establish the validity of estimate (6.19).

As a first consequence of estimates (6.18) and (6.19) we see that

$$\sup_{|x|<\delta} \left\{ | R_0^{\pm}(\mu_\pm)(x, y) P_2 g(y) | \, dy \right\} = O(1) \, \| P_2 g \|_{\mathfrak{B}} \, . \qquad (6.20)$$

Insertion of this fact in inequality $(6.12)_{12}$ yields the validity of conclusion $(6.9)_{12}$.

The validity of conclusion $(6.10)_{12}$ follows from an adaptation of the arguments used to establish $(6.9)_{12}$. Since this adaptation is similar to the one in the proof of Proposition $(6.1)_1$ we do not carry out the details, and consider the proof of Proposition $(6.1)_{12}$ complete.

Another adaptation of the arguments used to establish conclusion $(6.9)_{12}$, that we shall not carry out either, shows that Proposition $(6.1)_{12}$ remains valid if we interchange the roles of $P_1 f$ and $P_2 g$.

Finally, inserting this fact and Propositions $(6.1)_{(11),(22)}$ and $(6.1)_{12}$ in equation (6.11) we arrive at the validity of Lemma 6.1.

b. Condition $G_2(\mathscr{I})$. To verify the condition for each $\omega_\pm$ in the closure $\overline{\mathscr{R}_\pm(\mathscr{I})}$ we define the operator $(M(p) \, R_0^{\pm}(\omega_\pm))_{\mathfrak{B}}$ by the kernel,

$$(M(p) \, R_0^{\pm}(\omega_\pm))_{\mathfrak{B}} \, (x, y) = p(x) \, R_0^{\pm}(\omega_\pm)(x, y).$$

The fact that this kernel defines a bounded operator on $\mathfrak{B}$ is implied by the lemma that follows.

LEMMA 6.2. *Let the space $\mathfrak{B}$ be defined by the norm* (6.1) *and let $\mathscr{I}$ be a compact interval which does not contain zero. Suppose that the potential p_1 has bounded support, it satisfies Condition* I, *and that p_2 satisfies Condition* II *and Condition* K. *Then there is a constant $O(1)$ such that for every $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and f in $\mathfrak{B} \cap \mathfrak{L}_2(\mathscr{E}_d)$*

$$\| M(p_1 + p_2) R_0^\pm(\mu_\pm) f \|_{\mathfrak{B}} = O(1) \| f \|_{\mathfrak{B}}, \tag{6.21}$$

and for each $\omega_\pm$ in the closure $\overline{\mathscr{R}_\pm(\mathscr{I})}$,

$$\lim_{\mu_\pm = \omega_\pm} \| M(p_1 + p_2) R_0^\pm(\mu_\pm) - M(p_1 + p_2) R_0^\pm(\omega_\pm) \|_{\mathfrak{B}} = 0, \tag{6.22}$$

uniformly in $\omega_\pm$.

To verify conclusion (6.21) we first maintain that there is a constant $O(1)$ such that for every $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and f in $\mathfrak{B}$

$$\| P_1(M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} f \|_1 = O(1) \| f \|_{\mathfrak{B}}. \tag*{(6.21)$_1$}$$

To see this, for each $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and g in $\mathfrak{B}$ set

$$I_1(\mu_\pm, g) = \sup_{|x| < \delta} \int | R_0^\pm(\mu_\pm)(x, y) | \frac{| g(y) |}{\| g \|_{\mathfrak{B}}} dy. \tag*{(6.23)$_1$}$$

Since the supports of the potentials p_1 and p_2 are disjoint, and the support of p_1 is contained in the ball $\mathscr{B}_\delta$, we see from definitions $(6.1)_1$ and $(6.4)_1$ that

$$\| P_1(M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} f \|_1 \leqslant I_1(\mu_\pm, P_1 f) \| P_1 f \|_{\mathfrak{B}} + I_1(\mu_\pm, P_2 f) \| P_2 f \|_{\mathfrak{B}}. \tag*{(6.24)$_1$}$$

Estimates (6.13) and (6.20) show that

$$I_1(\mu_\pm, P_1 f) < \infty \qquad \textit{and} \qquad I_1(\mu_\pm, P_2 f) < \infty, \tag*{(6.25)$_{(11),(12)}$}$$

uniformly in $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and f in $\mathfrak{B}$. Insertion of these facts in $(6.24)_1$ yields the validity of conclusion $(6.21)_1$.

Second we maintain that,

$$\| P_2(M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} f \|_2 = O(1) \| f \|_{\mathfrak{B}}. \tag*{(6.21)$_2$}$$

To see this, set

$$I_2(\mu_\pm, g) = \sup_\xi \frac{1}{w_d(\xi)} \cdot \frac{| TR_0^\pm(\mu_+) g(\xi) |_{\mathfrak{A}}}{\| g \|_{\mathfrak{B}}}. \tag{6.23$_2$}$$

Then it is clear from definitions $(6.1)_2$, (2.1) and (5.1) that for every f,

$$\| P_2(M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} f \|_2 \leqslant I_2(\mu_\pm, P_1 f) \| P_1 f \|_{\mathfrak{B}} + I_2(\mu_\pm, P_2 f) \| P_2 f \|_{\mathfrak{B}}. \tag{6.24$_2$}$$

According to definitions $(6.1)_1$ and (5.5)

$$| TR_0^\pm(\mu_\pm) P_1 f(\xi) |_{\mathfrak{A}} = O(1) \| P_1 f \|_1 \xi^{(d-1)/2} \sup_{|x|=\xi} \int | R_0^\pm(\mu_\pm)(x, y) p_1(y) | \, dy. \tag{6.26}$$

This estimate together with estimate (6.13) and relations $(6.2)_{0,\infty}$ shows that

$$I_2(\mu_\pm, P_1 f) < \infty, \tag{6.25$_{21}$}$$

uniformly in $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and f in $\mathfrak{B}$. To estimate the second term in $(6.24)_2$ recall definition (5.5) which shows that

$$TR_0^\pm(\mu_\pm) P_2 f = TR_0^\pm T^* T P_2 f.$$

Hence, according to relation (5.7),

$$| TR_0^\pm(\mu_\pm) P_2 f(\xi) |_{\mathfrak{A}} \leqslant \int | TR_0^\pm(\mu_\pm) T^*(\xi, \eta) |_{\mathfrak{A}} \cdot | TP_2 f(\eta) |_{\mathfrak{A}} \, d\eta.$$

For brevity assume that $d \geqslant 3$. Then using Theorem 5.1 to estimate the first factor in this integral and definition $(6.1)_2$ to estimate the second factor, we obtain

$$\frac{1}{w_d(\xi)} | TR_0^\pm(\mu_\pm) P_2 f(\xi) |_{\mathfrak{A}} = O(1) \| P_2 f \|_2 \int \frac{\min(\xi, \eta^{1/3})}{w_d(\xi)} p_2^*(\eta) w_d(\eta) \, d\eta. \tag{6.27}$$

Relations $(6.2)_{0,\infty}$ show that there is a constant $O(1)$, such that for every η

$$\frac{\min(\xi, \eta^{1/3})}{w_d(\xi)} = O(1) \max(1, \eta^{1/3}).$$

According to relation (6.3)

$$\int \max(1, \eta^{1/3}) p_2^*(\eta) w_d(\eta) \, d\eta < \infty.$$

Inserting these facts in (6.27) we arrive at

$$I_2(\mu_\pm, P_2 f) < \infty, \tag{6.25$_{22}$}$$

uniformly in $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and f in $\mathfrak{B}$. This estimate, in turn, together with $(6.25)_{21}$ inserted in inequality $(6.24)_2$ yields the validity of conclusion $(6.21)_2$.

Third, we maintain that

$$\| (M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} f \|_3 = O(1) \| f \|_{\mathfrak{B}} . \tag{6.21$_3$}$$

To see this, set

$$I_3(\mu_\pm, g) = \sup_x \int | R_0^\pm(\mu_\pm)(x, y) | \frac{| g(y) |}{\| g \|_{\mathfrak{B}}} dy, \tag{6.23$_3$}$$

where the supremum is taken over all of $\mathscr{E}_d$. Then remembering definition $(6.1)_3$ we see that

$$\begin{aligned} \|(M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} f \|_3 \leqslant \| p_1 + p_2 \|_3 (I_3(\mu_\pm, P_1 f) \| P_1 f \|_{\mathfrak{B}} \\ + I_3(\mu_\pm, P_2 f) \| P_2 f \|_{\mathfrak{B}} . \end{aligned} \tag{6.24$_3$}$$

Estimate (6.13) shows that

$$I_3(\mu_\pm, P_1 f) < \infty, \tag{6.25$_{31}$}$$

uniformly in $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$ and f in $\mathfrak{B}$. In case $\delta = 0$, conclusion $(6.21)_3$ is evident, and so assume that $\delta \neq 0$. We claim that to δ there is a constant $O(1)$, such that

$$\sup_x \left\{ \int_{4\delta < 2|x-y| < |x| + \delta} | R_0^t(\mu_t)(x, y) P_2 f(y) | \, dy \right\} = O(1) \| P_2 f \|_2 . \tag{6.28}$$

Since $\delta \neq 0$ estimate $(6.8)_\infty$ together with the Schwarz inequality shows that the left member is majorized by

$$O(1) \left(\int_{2|x-y| < |x| + \delta} \left(\frac{1}{| x - y |} \right)^{d-1} p_2^*(| y |) \, dy \right)^{1/2} \cdot \left(\int \frac{| P_2 f(y) |^2}{p_2^*(| y |)} dy \right)^{1/2} .$$

Carrying out the spherical integration in the second factor and using definitions (5.5) and $(6.1)_2$ we obtain

$$\left(\int \frac{| P_2 f(y) |^2}{p_2^*(| y |)} dy \right)^{1/2} = O(1) \cdot \| P_2 f_2 \|_2 \cdot \left(\int_0^\infty p_2^*(\eta) \, w_d^2(\eta) \, d\eta \right)^{1/2} .$$

Since the support of $p_2^*(\eta)$ is disjoint from the interval $[0, \delta)$ and $\delta \neq 0$, we see from relation (6.3) that this integral is finite. An elementary application of the triangle inequality shows that for every vector y in $\mathscr{E}_d$

$$4\delta < 2 \mid x - y \mid < \mid x \mid + \delta \qquad \textit{implies} \qquad \mid y \mid > \tfrac{1}{6}(\mid x \mid + \delta).$$

Hence according to Condition K, for such values of y,

$$p^*(\mid y \mid) = O\left(\frac{1}{\mid x \mid + \delta}\right).$$

From these estimates and from the translation invariance of the Lebesgue measure, we obtain the validity of estimate (6.28). Estimates (6.28), (6.18), and (6.19) together show that

$$I_3(\mu_\pm\,, P_2 f) < \infty. \qquad (6.25)_{32}$$

Inserting estimates $(6.25)_{(31),(32)}$ in inequality $(6.24)_3$ yields the validity of conclusion $(6.21)_3$.

Finally combining estimates $(6.21)_{(1,2,3)}$ we arrive at the validity of conclusion (6.21).

To verify conclusion (6.22) we first maintain that

$$\lim_{\mu_\pm = \omega_\pm} \| P_1(M(p_1 + p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - P_1(M(p_1 + p_2)\, R_0^\pm(\omega_\pm))_{\mathfrak{B}} \|_1 = 0, \quad (6.22)_1$$

uniformly in $\omega_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$. To see this, for each integer k and vector g in $\mathfrak{B}$ set

$$I_1(\mu_\pm\,, \omega_\pm\,, g, k) = \sup_{|x|<\delta} \left\{\int_{|y|\leqslant k} \mid (R_0^\pm(\mu_\pm) - R_0^\pm(\omega_\pm)\,(x, y) \mid \frac{\mid g(y) \mid}{\| g \|_{\mathfrak{B}}}\, dy\right\}. \qquad (6.29)_1$$

Since the supports of the potentials p_1 and p_2 are disjoint, we see from definitions $(6.1)_1$ and $(6.4)_{1,2}$ that for every f in $\mathfrak{B}$

$$\| P_1[(M(p_1 + p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1 + p_2)\, R_0^\pm(\omega_\pm))_{\mathfrak{B}}] \|_1$$
$$\leqslant I_1(\mu_\pm\,, \omega_\pm\,, P_1 f, \infty) \| P_1 f \|_{\mathfrak{B}} + I_1(\mu_\pm\,, \omega_\pm\,, P_2 f, \infty) \| P_2 f \|_{\mathfrak{B}}. \qquad (6.30)_1$$

he arguments leading to estimates (6.13) and (6.20) show that

$$\lim_{k=\infty} I_1(\mu_\pm\,, \omega_\pm\,, P_{1,2} f, k) = I_1(\mu_\pm\,, \omega_\pm\,, P_{1,2} f, \infty), \qquad (6.31)_{(11),(12)}$$

and that this is uniform in $\mu_\pm$ in $\mathscr{R}_\pm(\mathscr{I})$, $\omega_\pm$ in $\overline{\mathscr{R}_\pm(\mathscr{I})}$ and f in $\mathfrak{B}$. Combining these arguments with the continuity relations of equation (6.7) and Theorem 5.2, we obtain, for frozen k and f

$$\lim_{\mu_\pm = \omega_\pm} I_1(\mu_\pm\,, \omega_\pm,\, P_{1,2} f, k) = 0. \qquad (6.32)_{(11),(12)}$$

These two relations together show [B.7.a] that

$$\lim_{\mu_\pm = \omega_\pm} I_1(\mu_\pm, \omega_\pm, P_{1,2}f, \infty) = 0. \qquad (6.33)_{(11),(12)}$$

At the same time it follows that this is uniform in $\omega_\pm$ in $\overline{\mathscr{R}_\pm(\mathscr{I})}$ and f in $\mathfrak{B}$. Insertion of this relation in inequality $(6.30)_1$ yields the validity of conclusion $(6.22)_1$.

Second, we maintain that

$$\lim_{\mu_\pm = \omega_\pm} \| P_2[(M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1 + p_2) R_0^\pm(\omega_\pm))_{\mathfrak{B}}] \|_2 = 0, \qquad (6.22)_2$$

uniformly in $\omega_\pm$ in $\overline{\mathscr{R}_\pm(\mathscr{I})}$. To see this, set

$$I_2(\mu_\pm, \omega_\pm, g, k) = \sup_{(1/k) < \xi < k} \left[\frac{1}{w_d(\xi)} \cdot \frac{| TR_0^\pm(\mu_\pm) g(\xi) - TR_0^\pm(\omega_\pm) g(\xi) |_{\mathfrak{A}}}{\| g \|_{\mathfrak{B}}}\right]. \qquad (6.29)_2$$

Then, similarly to inequality $(6.24)_2$, we have

$$\begin{aligned} &\| P_2[(M(p_1 + p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1 + p_2) R_0^\pm(\omega_\pm))_{\mathfrak{B}}] f \|_2 \\ &\leqslant I_2(\mu_\pm, \omega_\pm, P_1 f, \infty) \| P_1 f \|_{\mathfrak{B}} + I_2(\mu_\pm, \omega_\pm, P_2 f, \infty) \| P_2 f \|_{\mathfrak{B}}. \end{aligned} \qquad (6.30)_2$$

The arguments leading to estimate (6.26) together with relations $(6.2)_{0,\infty}$ show that

$$\lim_{k=\infty} I_2(\mu_\pm, \omega_\pm, P_1 f, k) = I_2(\mu_\pm, \omega_\pm, P_1 f, \infty), \qquad (6.31)_{21}$$

uniformly in $\mu_\pm$ and $\omega_\pm$ and f. Combining these arguments with the continuity relation (6.7) and estimates $(6.8)_{0,\infty}$ we obtain, for frozen k,

$$\lim_{\mu_\pm = \omega_\pm} I_2(\mu_\pm, \omega_\pm, P_1 f, k) = 0. \qquad (6.32)_{21}$$

Thus

$$\lim_{\mu_\pm = \omega_\pm} I_2(\mu_\pm, \omega_\pm, P_1 f, \infty) = 0. \qquad (6.33)_{21}$$

A slight adaptation of the arguments leading to estimate $(6.25)_{22}$ shows that estimate (6.27) implies

$$\lim_{k=\infty} I_2(\mu_\pm, \omega_\pm, P_2 f, k) = I_2(\mu_\pm, \omega_\pm, P_2 f, \infty). \qquad (6.31)_{22}$$

Combining these arguments with the continuity statement of Theorem 5.2 we obtain, for frozen k,

$$\lim_{\mu_\pm = \omega_\pm} I_2(\mu_\pm, \omega_\pm, P_2 f, k) = 0. \qquad (6.32)_{22}$$

Thus

$$\lim_{\mu_\pm=\omega_\pm} I_2(\mu_\pm, \omega_\pm, P_2 f, \infty) = 0. \tag{6.33$_{22}$}$$

Insertion of relations $(6.31)_{(21),(22)}$ in inequality $(6.30)_2$ yields the validity of conclusion $(6.22)_2$.

Third, we maintain that

$$\lim_{\mu_\pm=\omega_\pm} \| (M(p_1+p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1+p_2) R_0^\pm(\omega_\pm))_{\mathfrak{B}} \|_3 = 0. \tag{6.22$_3$}$$

To see this set

$$I_3(\mu_\pm, \omega_\pm, g, k) = \sup_{|x|<k} | R_0^\pm(\mu_\pm) g(x) - R_0^\pm(\omega_\pm) g(x) | \frac{1}{\| g \|_{\mathfrak{B}}}. \tag{6.29$_3$}$$

Then we see from definition $(6.1)_3$ that for every f in $\mathfrak{B}$,

$$\begin{aligned} &\| [(M(p_1+p_2) R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1+p_2) R_0^\pm(\omega_\pm))_{\mathfrak{B}}] f \|_3 \\ &\leqslant \| p_1 + p_2 \|_3 [I_3(\mu_\pm, \omega_\pm, P_1 f, \infty) \| P_1 f \|_{\mathfrak{B}} + I_3(\mu_\pm, \omega_\pm, P_2 f, \infty) \| P_2 f \|_{\mathfrak{B}}]. \end{aligned} \tag{6.30$_3$}$$

Definitions $(6.29)_{(1,3)}$ show that for $k = \delta$,

$$I_3(\mu_\pm, \omega_\pm, g, \delta) = I_1(\mu_\pm, \omega_\pm, g, \infty).$$

This relation together with $(6.33)_{(11),(12)}$ yields

$$\lim_{\mu_\pm=\omega_\pm} I_3(\mu_\pm, \omega_\pm, P_{1,2} f, k) = 0. \tag{6.32$_{(31),(32)}$}$$

We see from estimate (6.13) that

$$\lim_{k=\infty} I_3(\mu_\pm, \omega_\pm, P_1 f, k) = I_3(\mu_\pm, \omega_\pm, P_1 f, \infty). \tag{6.32$_{31}$}$$

The arguments leading to estimate $(6.25)_{32}$ show that

$$\lim_{k=\infty} I_3(\mu_\pm, \omega_\pm, P_2 f, k) = I_3(\mu_\pm, \omega_\pm, P_2 f, \infty). \tag{6.32$_{32}$}$$

Since both of these limit relations are uniform in $\mu_\pm$ and $\omega_\pm$ and f, we obtain

$$\lim_{\mu_\pm=\omega_\pm} I_3(\mu_\pm, \omega_\pm, P_{1,2} f, \infty) = 0. \tag{6.33$_{(31),(32)}$}$$

Insertion of these relations in inequality $(6.30)_3$ yields the validity of conclusion $(6.22)_3$.

Finally, combining conclusions $(6.22)_{(1,2,3)}$ we arrive at the validity of conclusion (6.22). This completes the proof of Lemma 6.2.

(c) CONDITION $G_3(\mathscr{I})$. To verify this condition for each integer k set

$$p_2^{(k)}(x) = \begin{cases} p_2(x) & \dfrac{1}{k} < |x| < k \\ 0 & \text{otherwise.} \end{cases} \tag{6.34}^{(k)}$$

The lemma that follows implies the validity of this condition.

LEMMA 6.3. *Let the space* $\mathfrak{B}$ *be defined by relation* (6.1) *and let* $\operatorname{Im} \mu_\pm \neq 0$. *Suppose that the potentials* p_1 *and* p_2 *satisfy the assumptions of Theorem 2.1. Then for each k the operator* $M(p_1 + p_2^{(k)})$ *is* Δ*-compact with reference to the* $\mathfrak{L}_2(\mathscr{E}_d)$ *norm, and*

$$\lim_{k=\infty} \| (M(p_1 + p_2^{(k)})\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1 + p_2^{(k)})\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} \| = 0. \tag{6.35}$$

Furthermore the operators $(M(p_1 + p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}}$ *are compact.*

To verify the first conclusion recall that according to the assumptions of Theorem 2.1 the operator $M(p_1)$ is Δ-compact with reference to the $\mathfrak{L}_2(\mathscr{E}_d)$-norm. Condition II together with estimate (A.1) of the Appendix shows that

$$\lim_{\delta=0} \left[\sup_x \int_{|x-y|<\delta} | p_2{}^k(y) |^2 \begin{Bmatrix} 1, & d \leqslant 3 \\ \left(\dfrac{1}{x-y}\right)^{d-4} & d \geqslant 4 \end{Bmatrix} dy \right] = 0.$$

This, in turn, shows [B.4], [B.5] that each of the $M(p_2^{(k)})$ is Δ-compact with reference to the $\mathfrak{L}_2(\mathscr{E}_d)$ norm. That is, the first conclusion holds.

To verify conclusion (6.35) first recall that by assumption the supports of p_1 and p_2, hence p_1 and $p_2^{(k)}$, are disjoint. This yields, for each k,

$$\| P_1[(M(p_1 + p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1 + p_2^{(k)})\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} \|_1 = 0. \tag{6.35$_1$}$$

We claim that

$$\lim_{k=\infty} \| P_2[(M(p_1 + p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1 + p_2^{(k)})\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} \|_2 = 0. \tag{6.35$_2$}$$

To see this, in analogy to definition $(6.29)_2$ set

$$I_2(\mu_\pm, g, k) = \sup_{(1/k)<\xi<k} \left[\frac{1}{w_d(\xi)} \frac{|TR_0^\pm(\mu_\pm)\, g(\xi)|_{\mathfrak{A}}}{\|g\|_{\mathfrak{B}}}\right].$$

Then in analogy to inequality $(6.30)_2$ we have

$$\begin{aligned} &\| P_2[(M(p_1+p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1+p_2^{(k)})\, R_0^\pm(\mu_\pm))_{\mathfrak{B}}] f\|_2 \\ &\leqslant [I_2(\mu_\pm, P_1 f, \infty) - I_2(\mu_\pm, P_1 f, k)]\, \|P_1 f\|_{\mathfrak{B}} \\ &\qquad + [I_2(\mu_\pm, P_2 f, \infty) - I_2(\mu_\pm, P_2 f, k)]\, \|P_2 f\|_{\mathfrak{B}}. \qquad (6.36)_2 \end{aligned}$$

The arguments leading to relations $(6.31)_{(21),(22)}$ show that

$$\lim_{k=\infty} I_2(\mu_\pm, P_{1,2} f, k) = I_2(\mu_\pm, P_{1,2} f, \infty),$$

uniformly in f in $\mathfrak{B}$. Insertion of this fact in inequality $(6.36)_2$ yields the validity of conclusion $(6.35)_2$.

Next we claim that

$$\lim \| (M(p_1+p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1+p_2^{(k)})\, R_0^\pm(\mu_\pm)_{\mathfrak{B}} \|_3 = 0. \qquad (6.35)_3$$

For, in analogy to inequality $(6.24)_3$ we have, for every f in $\mathfrak{B}$

$$\begin{aligned} &\| [(M(p_1+p_2)\, R_0^\pm(\mu_\pm))_{\mathfrak{B}} - (M(p_1+p_2^{(k)})\, R_0^\pm(\mu_\pm))_{\mathfrak{B}}] f\|_3 \\ &\leqslant \|p_2 - p_2^{(k)}\|_3 \cdot (I_3(\mu_\pm, P_1 f)\, \|P_1 f\|_{\mathfrak{B}} + I_3(\mu_\pm, P_2 f)\, \|P_2 f\|_{\mathfrak{B}}). \qquad (6.37) \end{aligned}$$

For brevity assume that $d \geqslant 3$. Insertion of definition $(6.35)^{(k)}$ in estimate $(A.1)_d$ of the Appendix, yields

$$\int_{|x-y|<2\delta} |p_2(y) - p_2^{(k)}(y)| \left(\frac{1}{|x-y|}\right)^{d-2} dy = O(1) \min\left(1, \frac{1}{|x|}\right) \int p_2^*(\eta)\, \eta\, d\eta,$$

where the integral on the right is extended over

$$|x| - 2\delta < \eta < |x| + 2\delta \qquad \text{and} \qquad 0 < \eta < \frac{1}{k} \quad \text{or} \quad \eta > k.$$

Hence we see from Condition II and definition $(6.1)_3$ that

$$\lim_{k=\infty} \|p_2 - p_2^{(k)}\|_3 = 0.$$

Insertion of this fact and of estimates $(6.25)_{(31),(32)}$ in inequality (6.37) yields the validity of conclusion $(6.35)_3$.

Having established conclusions $(6.35)_{(1,2)}$ and $(6.35)_3$ we arrive at the validity of conclusion (6.35).

According to this conclusion, to verify the compactness of $(M(p_1 + p_2) R_0^{\pm}(\mu_{\pm}))_{\mathfrak{B}}$ it suffices to show that for each k the operators $(M(p_1 + p_2^{(k)}) R_0^{\pm}(\mu_{\pm}))_{\mathfrak{B}}$ are compact. To see this, let $\mathfrak{C}(\mathscr{B}_k)$ denote the space of continuous functions on the ball $\mathscr{B}_k$ in $\mathscr{E}_d$, with the maximum norm. The arguments used to establish conclusion (6.22) show that the transformations

$$R_0^{\pm}(\mu_{\pm}) : \mathfrak{B} \to \mathfrak{C}(\mathscr{B}_k)$$

are continuous. At the same time we see that a bounded subset of $\mathfrak{B}$ is mapped into an equicontinuous and uniformly bounded set of functions. Hence according to the Arzela-Ascoli criterion [B.7.b] this map is compact. Definitions (6.1) and $(6.34)^{(k)}$ show that the transformation

$$M(p_1 + p_2^{(k)}) : \mathfrak{C}(\mathscr{B}_k) \to \mathfrak{B},$$

is bounded. Thus, each of the operators $(M(p_1 + p_2^{(k)}) R_0^{\pm}(\mu_{\pm}))_{\mathfrak{B}}$ can be factored as a product of a bounded and a compact transformation. This establishes the compactness of these operators and the proof of Lemma 5.3 is complete.

(d) CONDITION $A_1(\mathscr{I})$. Let $\mathscr{I}$ be a compact interval of the positive axis which does not contain zero. In this subsection we show that if the potential $p = p_1 + p_2$ satisfies the assumptions of Theorem 2.1 then Condition $A_1(\mathscr{I})$ holds. That is to say, for each ω in $\mathscr{I}$ the operators $(1 - M(p_1 + p_2) R_0^{\pm}(\omega))_{\mathfrak{B}}$ are invertible. According to Lemma 6.3 the second term is compact and hence according to the Fredholm alternative [B.7.c] it suffices to show that this operator is one to one. In other words assume that $\tilde{\omega}$ is an exceptional point in $\mathscr{I}$ and $\tilde{f}$ is an exceptional vector in $\mathfrak{B}$ for which either

$$(1 - M(p_1 + p_2) R_0^{+}(\tilde{\omega}))_{\mathfrak{B}} \tilde{f} = 0 \quad \text{or} \quad (1 - M(p_1 + p_2) R_0^{-}(\tilde{\omega}))_{\mathfrak{B}} \tilde{f} = 0. \tag{6.38$^{\pm}$}$$

Then we have to show that

$$\tilde{f} = 0. \tag{6.39}$$

To verify this implication first note that the sesquilinear form $[R_0^{\pm}(\tilde{\omega})]_{\mathfrak{B}}$ and the vector $\tilde{f}$ define a functional on $\mathfrak{B}$. Namely, the functional which assigns to the vector g in $\mathfrak{B}$ the complex number $[R_0(\tilde{\omega})]_{\mathfrak{B}} (\tilde{f}, g)$. In view of our choice of $\mathfrak{B}$ this functional corresponds to the function

$$R_0^{\pm}(\tilde{\omega}) \tilde{f}(x) = \int R_0^{\pm}(\tilde{\omega}) (x, y) \tilde{f}(y) \, dy. \tag{6.40}$$

First consider the case of dimensions $d \geqslant 2$. As a consequence of assumption $(6.38)^{\pm}$ we show that this function is twice continuously differentiable near any point x at which $p(x)$ is Hölder continuous. We claim that $\tilde{f}$ is bounded near such a point x. To see this, recall estimate $(6.8)_\infty$ and definition (6.1). They imply that

$$\sup_x \left\{ \int_{|x-y|>1} | R_0^{\pm}(\tilde{\omega})(x, y) \tilde{f}(y) | \, dy \right\} < \infty.$$

Insertion of this estimate and of $(6.8)_0$ in assumption $(6.36)^{\pm}$ yields for $d \geqslant 3$,

$$\tilde{f}(x) = O(1) \int_{|x-y|<1} \left(\frac{1}{| x - y |} \right)^{d-2} \tilde{f}(y) \, dy + O(1), \qquad (6.41)_d$$

and for $d = 2$ a similar formula with a log-kernel. Note that for dimensions $d \leqslant 3$ these kernels are square integrable and according to definition (6.1) $\tilde{f}$ is locally square integrable. Hence in case of dimensions $d \leqslant 3$, the Schwarz inequality shows that $\tilde{f}$ is bounded near x. In case of dimensions $d \geqslant 4$ the Sobolev inequality [B.3] shows that near x it is integrable with power $\alpha(1)$, where

$$\frac{1}{\alpha(1)} = \frac{1}{2} + \frac{d-2}{d} - 1 = \frac{d-4}{d}.$$

Next define the integer n by the inequality

$$0 \leqslant d - 4n < 4.$$

Then a repeated application of the Sobolev inequality [B.3] shows that near x the function $\tilde{f}$ is integrable with power $\alpha(n)$ where

$$\frac{1}{\alpha(n)} = \frac{d-4n}{2d} \qquad \text{hence} \qquad \alpha(n) > \frac{d}{2}.$$

This lower estimate for $\alpha(n)$ allows us to apply the Hölder inequality to the left member of $(6.41)_d$ and we obtain the boundedness of $\tilde{f}$ near x, as we have claimed. Insertion of this boundedness in (6.40) allows us to differentiate under the integral sign. This fact in turn, inserted in assumption $(6.38)^{\pm}$ yields the Hölder continuity of $\tilde{f}(x)$ near x. As is well known [B.3], [B.12] this Hölder continuity implies that the right member of (6.40) is twice continuously differentiable as we have maintained. A similar elementary argument, that we shall not carry out, shows that this function satisfies the differential equation

$$- \Delta R_0^{\pm}(\tilde{\omega}) \tilde{f}(x) + p(x) R_0^{\pm}(\tilde{\omega}) \tilde{f}(x) = \tilde{\omega} \tilde{f}(x). \qquad (6.42)$$

As a second consequence of assumption $(6.38)^{\pm}$ we derive an asymptotic description of this function near $x = \infty$. To do this we need a spectral transformation of $-\Delta$. That is a unitary transformation mapping $\mathfrak{L}_2(\mathscr{E}_d)$ into an $\mathfrak{L}_2(0, \infty), \mathfrak{A})$ space which carries $-\Delta$ into the multiplication operator. As is well known for $-\Delta$ the accessory space $\mathfrak{A}$ can be chosen to be $\mathfrak{L}_2(\mathscr{S}_{d-1})$. Since Fourier transformation carries $-\Delta$ into the square of the multiplication operator we see that the kernel

$$U_0(\omega, y)(u) = \frac{1}{\sqrt{2}}\left(\frac{1}{2\pi}\right)^{n/2}\left(\frac{1}{\omega}\right)^{1/4}\omega^{(d-1)/2}\exp(i(u, y)\sqrt{\omega}),$$

$$\omega \in [0, \infty), \qquad y \in \mathscr{E}_d, \qquad u \in \mathscr{S}_{d-1} \tag{6.43}$$

defines such a transformation. It was observed elsewhere [A.13.a] that, under general circumstances, Condition $G_1(\mathscr{I})$ implies that U_0 can be extended to $\mathfrak{B}$, and it maps vectors in $\mathfrak{B}$ into continuous $\mathfrak{L}_2(\mathscr{S}_{d-1})$-valued functions. Hence for each ω in $\mathscr{I}$, the function value

$$U_0 f(\omega) \in \mathfrak{L}_2(\mathscr{S}_{d-1}), \qquad f \in \mathfrak{B},$$

is well defined. We shall also need the evident fact that the transformation T of definition (5.5) maps continuous functions on $\mathscr{E}_d$ into continuous $\mathfrak{L}_2(\mathscr{S}_{d-1})$ valued functions. With the aid of these notations we formulate,

LEMMA 6.4. *To each positive number ω there is a constant γ such that for every f in $\mathfrak{B}$,*

$$\lim_{\xi=\infty} | TR_0^{\pm}(\omega) f(\xi) |_{\mathfrak{A}} = \gamma \, | U_0 f(\omega) |_{\mathfrak{A}} . \tag{6.44}$$

To verify this lemma we need the asymptotic description of the functions $k^{\pm}$ of relation (6.6). It is well known that there is a constant γ such that

$$k^{\pm}(\omega, \rho) \sim \gamma \left(\frac{1}{\rho}\right)^{(d-1)/2} \exp(\pm i\rho\sqrt{\omega}) \qquad \text{at} \qquad \rho \sim \infty.$$

Hence

$$k^{\pm}(\omega, | \xi u - y |) \sim \gamma \left(\frac{1}{\xi}\right)^{(d-1)/2} \exp(\pm i\xi\sqrt{\omega}) \cdot \exp(\pm i(u, y)\sqrt{\omega})$$

$$\text{at} \qquad \xi \sim \infty.$$

At the same time it follows that this is uniform in y on any compact subset of $\mathscr{E}_d$ and in u in $\mathfrak{L}_2(\mathscr{S}_{d-1})$. Thus if f in $\mathfrak{B}$ has compact support then

$$\xi^{(d-1)/2} \int R_0^{\pm}(\omega)\,(\xi u, y) f(y)\, dy \sim \gamma \exp(\pm\, i(u, y) \sqrt{\omega}) f(y)\, dy$$
$$\text{near} \qquad \xi \sim \infty.$$

Since this holds uniformly in u in $\mathfrak{L}_2(\mathscr{S}_{d-1})$, this implies that the u-integral of the absolute value squares are also asymptotic. Hence remembering definitions (5.1), (5.5) and relation (6.43) we arrive at

$$|\, TR_0^{\pm}(\omega) f(\xi)\, |_{\mathfrak{A}} \sim |\, U_0 f(\omega)\, |_{\mathfrak{A}}\,, \qquad \text{near} \qquad \xi \sim \infty.$$

That is to say, conclusion (6.44) holds under the additional assumption that f has bounded support.

To verify this conclusion for a general f, set

$$f_k(\eta) = \begin{cases} f(\eta) & 0 < \eta \leqslant k \\ 0 & k < \eta \end{cases}.$$

Then we see from definition (6.1) and estimate (A.1) of the Appendix, that

$$\lim_{k=\infty} \| f - f_k \|_{\mathfrak{B}} = 0. \tag{6.45}$$

This fact together with estimates $(6.25)_{(21),(22)}$ yields, at each point ξ,

$$\lim_{k=\infty} |\, TR_0^{\pm}(\omega) f_k(\xi)\, |_{\mathfrak{A}} = |\, TR_0^{\pm}(\omega) f(\xi)\, |_{\mathfrak{A}}\,.$$

At the same time it follows that this is uniform in ξ in $[0, \infty)$. According to the already established part of the lemma

$$\lim_{\xi=\infty} |\, TR_0^{\pm}(\omega) f_k(\xi)\, |_{\mathfrak{A}} = \gamma\, |\, U_0 f_k(\omega)\, |_{\mathfrak{A}}\,.$$

As is well known, these two relations together imply

$$\lim_{\xi=\infty} |\, TR_0^{\pm}(\omega) f(\xi)\, |_{\mathfrak{A}} = \lim_{k=\infty} \gamma\, |\, U_0 f_k(\omega)\, |_{\mathfrak{A}}\,.$$

According to an observation made elsewhere [A.13.a], under general circumstances the product of the evaluation functional with U_0 is a bounded functional on $\mathfrak{B}$. Hence remembering relation (6.45) we obtain

$$\lim_{k=\infty} |\, U_0 f_k(\omega)\, |_{\mathfrak{A}} = |\, U_0 f(\omega)\, |\,.$$

Upon inserting this equation in the previous one we arrive at the validity of conclusion (6.44). This completes the proof of Lemma 6.4.

Next we apply Lemma 6.4 to the exceptional point $\tilde{\omega}$ and vector $\tilde{f}$ of assumption (6.40). This yields

$$\lim_{\xi=\infty} | TR_0^{\pm}(\tilde{\omega}) f(\xi) |_{\mathfrak{A}} = | U_0 \tilde{f}(\tilde{\omega}) |_{\mathfrak{A}} . \tag{6.44}$$

According to the basic Lemma 3.1 of [A.13], assumption (6.40) implies that the right member is zero. Hence

$$\lim_{\xi=\infty} | TR_0^{\pm}(\tilde{\omega}) f(\xi) |_{\mathfrak{A}} = 0. \tag{6.46}$$

Now we invoke a theorem of Kato [B.1]. This says that if the potential p satisfies his Condition K then each solution of equation (6.42) for which (6.44) holds, vanishes in some neighborhood of infinity. Hence remembering the unique continuation principle [B.3] we obtain that for every x in $\mathscr{E}_d$

$$R_0^{\pm}(\tilde{\omega}) \tilde{f}(x) = 0.$$

Finally inserting this relation in assumption $(6.38)^{\pm}$ we arrive at the validity of conclusion (6.39) in case of dimensions $d \geqslant 2$.

Next consider the case of dimension $d = 1$. In this case the integrability assumption on the potential p does not imply that $R_0^{\pm}(\tilde{\omega}) \tilde{f}$ is twice continuously differentiable. Nevertheless it does imply that it has an absolutely continuous first derivative. Then from an extended Titchmarsh lemma [B.7.g] we can conclude the validity of conclusion (6.39). Since this was carried out elsewhere [A.14.b] for a related operator we skip the details.

Thus conclusion (6.39) holds for arbitrary dimensions and this establishes the validity of Condition $A_1(\mathscr{I})$ for the pair of operators

$$(-\Delta, (-\Delta + M(p_1 + p_2))_F).$$

(e) CONDITION $A_2(\mathscr{I})$. Let T be a bounded operator acting on an abstract Banach space $\mathfrak{B}$ and suppose that $1 - T$ is invertible. Then clearly

$$(1 - T)^{-1} - 1 = (1 - T)^{-1} T.$$

The already established Conditions $G_1(\mathscr{I})$ and $A_1(\mathscr{I})$ show that if we choose $\mu_{\pm}$ in $R_{\pm}(\mathscr{I})$ close enough to the interval $\mathscr{I}$, then the operator

$$T = (M(p_1 + p_2) R_0^{\pm}(\mu_{\pm}))_{\mathfrak{B}} ,$$

has this property. Hence Condition $A_2(\mathscr{I})$ is implied by the relation

$$[R_1^{\pm}(\mu_{\pm})]_{\mathfrak{B}} = [R_0^{\pm}(\mu_{\pm})]_{\mathfrak{B}} (1 - M(p_1 + p_2) R_0^{\pm}(\mu_{\pm}))_{\mathfrak{B}}^{-1}. \tag{6.47}$$

To verify this relation, recall the family of approximating potentials introduced in $(6.34)^k$. According to considerations used by Schechter [B.5] and elsewhere [B.4], there is a constant $O(1)$ such that for every k, in the $\mathfrak{H} = \mathfrak{L}_2(\mathscr{E}_d)$ operator norm,

$$\| M(| p_2^{(k)} |^{1/2}) R_0^{\pm}(\mu_{\pm}) M(| p_2^{(k)} |^{1/2}) \|_{\mathfrak{H}} = O(1) I_{d,1}(p_2^{(k)}).$$

This implies that

$$\begin{aligned} \| (R_0^{\pm}(\mu_{\pm}))^{1/2} M(p_1 + p_2^{(k)}) (R_0^{\pm}(\mu_{\pm}))^{1/2} - (R_0^{\pm}(\mu_{\pm}))^{1/2} M(p_1 + p_2) (R_0^{\pm}(\mu_{\pm}))^{1/2} \|_{\mathfrak{H}} \\ = O(1) I_{d,1}(p_2 - p_2^{(k)}). \end{aligned} \tag{6.48}$$

Definitions (2.2) and $(6.34)^{(k)}$ together with estimate (A.1) show that the right member converges to zero. Hence, we have in the $\mathfrak{L}_2(\mathscr{E}_d)$ operator norm,

$$\begin{aligned} \lim_{k=\infty} (R_0^{\pm}(\mu_{\pm}))^{1/2} (M(p_1 + p_2^{(k)}) (R_0^{\pm}(\mu_{\pm}))^{1/2} \\ = (R_0^{\pm}(\mu_{\pm}))^{1/2} M(p_1 + p_2) (R_0^{\pm}(\mu_{\pm}))^{1/2}. \end{aligned}$$

According to a lemma formulated elsewhere [B.4.a], $R_1^{\pm}(\mu_{\pm})$, the resolvent of $(-\Delta + M(p_1 + p_2))_F$, can be expressed in terms of $R_0(\mu_{\pm})$ as

$$\begin{aligned} R_1^{\pm}(\mu_{\pm}) = (R_0^{\pm}(\mu_{\pm}))^{1/2} \cdot [1 - (R_0^{\pm}(\mu_{\pm}))^{1/2} M(p_1 + p_2) (R_0^{\pm}(\mu_{\pm}))^{1/2}]^{-1} \\ \cdot (R_0^{\pm}(\mu_{\pm}))^{1/2}. \end{aligned} \tag{6.49}$$

These two equations together show that for large enough k

$$[1 - (R_0^{\pm}(\mu_{\pm}))^{1/2} M(p_1 + p_2) (R_0^{\pm}(\mu_{\pm}))] f = 0 \qquad \text{implies} \qquad f = 0.$$

According to Lemma 6.3, $M(p_1 + p_2^{(k)})$ is Δ-compact. Hence

$$\begin{aligned} [1 - (R_0^{\pm}(\mu_{\pm}))^{1/2} M(p_1 + p_2^{(k)}) (R_0^{\pm}(\mu_{\pm}))^{1/2}] \cdot (R_0^{\pm}(\mu_{\pm}))^{1/2} \\ = (R_0^{\pm}(\mu_{\pm}))^{1/2} \cdot (1 - M(p_1 + p_2^{(k)}) R_0^{\pm}(\mu_{\pm})). \end{aligned} \tag{6.50}$$

At the same time it follows that the second factor on the right is invertible. Hence (6.58) yields

$$\begin{aligned} (R_0^{\pm}(\mu_{\pm}))^{1/2} = [1 - (R_0^{\pm}(\mu_{\pm}))^{1/2} M(p_1 + p_2^{(k)}) (R_0^{\pm}(\mu_{\pm}))^{1/2}] \\ \cdot (R_0^{\pm}(\mu_{\pm}))^{1/2} \cdot (1 - M(p_1 + p_2^{(k)}) R_0^{\pm}(\mu_{\pm}))^{-1}. \end{aligned}$$

Finally from this equation we obtain

$$(R_0^{\pm}(\mu_{\pm}))^{1/2} \cdot [1 - R_0^{\pm}(\mu_{\pm})^{1/2} M(p_1 + p_2^{(k)}) (R_0^{\pm}(\mu_{\pm}))^{1/2}]^{-1} \cdot (R_0^{\pm}(\mu_{\pm}))^{1/2}$$
$$= R_0^{\pm}(\mu_{\pm}) \cdot (1 - M(p_1 + p_2^{(k)}) R_0^{\pm}(\mu_{\pm}))^{-1}.$$

Remembering equations (6.48) and (6.49) we see that as k tends to infinity the left member tends to $R_1^{\pm}(\mu_{\pm})$ in the $\mathfrak{L}_2(\mathscr{E}_d)$ operator-norm. The second factor on the right need not converge in the $\mathfrak{L}_2(\mathscr{E}_d)$ operator-norm. Nevertheless we see from conclusion (6.35) of Lemma 6.3 that the second factor converges in the $\mathfrak{B}$-operator-norm. Since the operators $R_0^{\pm}(\mu_{\pm})$ on $\mathfrak{L}_2(\mathscr{E}_d)$ determine bounded forms on $\mathfrak{B} \times \mathfrak{B}$ we arrive at the validity of relation (6.47). This establishes the validity of Condition $A_2(\mathscr{I})$.

Now let us summarize our statements. We have shown that the assumptions of Theorem 2.1 imply that the pair of operators $(-\Delta, (-\Delta + M(p_1 + p_2)_F)$, satisfy the assumptions of Theorem 3.1 over any compact interval $\mathscr{I}$ of the positive axis which does not contain zero. Therefore, the continuous parts of these operators over such an interval $\mathscr{I}$ are unitarily equivalent. Since the spectral projectors of a self-adjoint operator are countably additive [B.7], the continuous parts of these operators over the entire positive axis are unitarily equivalent. According to general considerations [B.4], [B.5], the compactness relation (4.2) implies that the essential spectrum of $(-\Delta + M(p_1 + p_2))_F$ is disjoint from the positive axis. Hence its continuous part over the positive axis equals its entire continuous part. Thus we arrive at the validity of Theorem 2.1.

Appendix

In this appendix we show that Condition II implies Condition I. Clearly this is true for dimension $d = 1$ and hence we can assume that $d \geqslant 2$. According to definition (2.2) it suffices to show that there is a constant $O(1)$ such that for every δ and x in $\mathscr{E}_2$

$$\int_{|x-y|<\delta} |p_2(y)| \log \frac{1}{|x-y|} \, dy$$
$$\leqslant O(1) \min\left(1, \frac{1 + |\log |x||}{|x|}\right) \int_{|x|-\delta}^{|x|+\delta} p_2^{\ast}(\eta)\, \eta(1 + |\log \eta|)\, d\eta \qquad (\text{A.1})_2$$

and x in $\mathscr{E}_d$ with $d \geqslant 3$

$$\int_{|x-y|<\delta} |p_2(y)| \left(\frac{1}{|x-y|}\right)_{dy}^{d-2} = O(1) \min\left(1, \frac{1}{|x|}\right) \int_{|x|-\delta}^{|x|+\delta} p_2^{\ast}(\eta)\, \eta\, d\eta. \quad (\text{A.1})_d$$

The integral on the right is extended for positive values of η only and for brevity this is not indicated explicitly.

To verify this estimate we need an extension of a well known integral formula [B.12.b]. This extension is formulated in the lemma that follows. In it σ_{d-1} denotes the volume of the $(d-1)$-dimensional unit ball and $\mathscr{S}(u, \alpha)$ denotes the cap of the $(d-1)$-dimensional unit sphere corresponding to angle α and center at the end-point of u. More specifically, we set

$$\mathscr{S}(u, \alpha) = \mathscr{E}[v : v \in \mathscr{E}_d , \mid v \mid = 1, \cos \alpha \leqslant (u, v) \leqslant 1].$$

LEMMA A. *Suppose that to the function $f(v)$ of the variable v in $\mathscr{E}_d$ there is a unit vector u and a function g such that for every v,*

$$f(v) = g((u, v)). \tag{A.2}$$

Then

$$\int_{\mathscr{S}(u,\alpha)} f(v)\, dS_v = (d-1)\, \sigma_{d-1} \int_{\cos\alpha}^{1} g(s)\, (1 - s^2)^{(d-3)/2}\, ds. \tag{A.3}$$

To verify conclusion (A.3) we introduce some notations. Let $\mathscr{C}_d(u, \alpha)$ denote the cone in $\mathscr{E}_d$ around the unit vector u with angle α. That is, set

$$\mathscr{C}_d(u, \alpha) = \mathscr{E}[v : v \in \mathscr{E}_d , \mid v \mid \cos \alpha \leqslant (u, v) \leqslant \mid v \mid]. \tag{A.4}$$

Indicentally we note that for α in $[0, \pi/2]$ this cone is convex and for α in $(\pi/2, \pi)$ its complement is convex. For $\alpha = \pi$ this cone is convex again since it equals $\mathscr{E}_d$. Let $\mathscr{B}_d(0, r)$ denote the ball in $\mathscr{E}_d$ of radius r with center at the origin and set

$$\mathscr{R}(r, u, \alpha) = \mathscr{B}_d(0, r) \cap \mathscr{C}_d(u, \alpha).$$

Then, remembering the definition of the spherical measure induced by the Lebesgue measure in $\mathscr{E}_d$, we have

$$\left[\frac{d}{dr} \int_{\mathscr{R}(r,u,\alpha)} f(y)\, dy\right]_{r=1} = \int_{\mathscr{S}(u,\alpha)} f(v)\, dS_v . \tag{A.5}$$

Next we use assumption (A.2) to evaluate the left member of (A.5). For brevity assume that α is in $[0, \pi/2]$. For each positive number s let $\mathscr{L}(su)$ denote the hyperplane in $\mathscr{E}_d$ which is perpendicular to u and goes through the point su. It is an elementary fact that

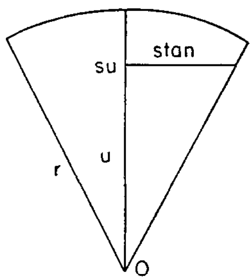
$$\mathscr{R}(r, u, \alpha) \cap \mathscr{L}(su) = \begin{cases} \mathscr{B}_{d-1}(su, s \tan \alpha), & 0 \leqslant s \leqslant r \cos \alpha \\ \mathscr{B}_{d-1}(su, (r^2 - s^2)^{1/2}), & r \cos \alpha \leqslant s \leqslant r, \end{cases}$$

and in case of dimension $d = 2$ this is illustrated in Fig. I. This relation together with assumption (A.2) yields

$$\int_{\mathscr{R}(r,u,\alpha)} f(y)\,dy = \sigma_{d-1} \int_0^{r\cos\alpha} g(s)\,(s\tan\alpha)^{d-1}\,ds + \sigma_{d-1} \int_{r\cos\alpha}^{r} g(s)\,(r^2 - s^2)^{(d-1)/2}\,ds,$$

since the level surfaces of the function f are the hyperplanes $\mathscr{L}(su)$. Differentiating this formula with respect to r and inserting the result in relation (A.5) we obtain the validity of conclusion (A.3). This completes the proof of Lemma A.

Fig. I



To derive conclusion (A.1) from Lemma A, for each pair of vectors x, y in $\mathscr{E}_d \times \mathscr{E}_d$ set

$$x = \xi u, \qquad u \in \mathscr{E}_d, \qquad |u| = 1$$
$$y = \eta v, \qquad v \in \mathscr{E}_d, \qquad |v| = 1,$$

and with the aid of a given positive number δ define

$$\alpha = \begin{cases} \arctan \dfrac{\delta}{\xi - \delta} & \xi > \delta \\ \pi & 0 \leqslant \xi < \delta. \end{cases} \tag{A.6}$$

We claim that the ball $\mathscr{B}_d(x, \delta)$ is contained in the intersection of the cone $\mathscr{C}_d(u, \alpha)$ and the annulus between the balls

$$\mathscr{B}_d(0, \xi + \delta) \qquad \text{and} \qquad \mathscr{B}_d(0, \max(0, \xi - \delta)).$$

For, in case of dimensions $d = 2$ this is evident from Fig. IIa and IIb. The case of dimensions $d \geqslant 3$ follows from this special case if we note that both the ball $\mathscr{B}_d(x, \delta)$ and the annulus are symmetric with respect to the axis determined by the unit vector u. This inclusion relation together with definition (2.2) shows that for $d = 2$

$$\int_{|x-y|\leqslant\delta} (1 + \log|x-y|)\, |p_2(y)|\, dy$$

$$\leqslant \int_{\xi-\delta}^{\xi+\delta} p_2^*(\eta)\, \eta \int_{\mathscr{S}(u,\alpha)} (1 + \log|\xi u - \eta v|)\, ds_v\, d\eta \qquad (A.7)_2$$

Fig. IIa

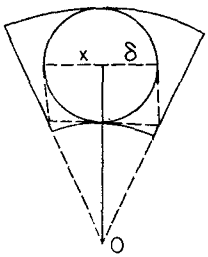
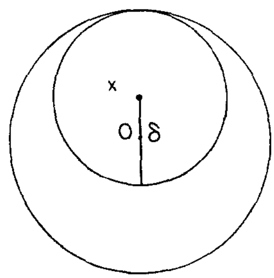


Fig. II b



and for $d \geqslant 3$,

$$\int_{|x-y|<\delta} \left(\frac{1}{|x-y|}\right)^{d-2} |p_2(y)|\, dy$$

$$\leqslant \int_{\xi-\delta}^{\xi+\delta} p_2^*(\eta)\, \eta^{d-1} \int_{\mathscr{S}(u,\alpha)} \left|\frac{1}{\xi u - \eta u}\right|^{d-2} ds_v\, d\eta. \qquad (A.7)_d$$

First we estimate the inner integral on the right for the case dimensions $d \geqslant 3$. To do this for frozen ξ, η and u set

$$f_d(v) = \left(\frac{1}{|\xi u - \eta v|}\right)^{d-2}.$$

Then defining the function $g_d(s)$ of the variable s by

$$g_d(s) = \left(\frac{1}{\xi^2 + \eta^2 - 2\xi\eta s}\right)^{(d-2)/2}, \qquad (A.8)_d$$

we have

$$f_d(v) = g_d(u, v)).$$

Hence application of Lemma A yields

$$\int_{\mathscr{S}(u,\alpha)} \left(\frac{1}{|\xi u - \eta v|}\right)^{d-2} ds_v = (d-1)\, \sigma_{d-1} \int_{\cos\alpha}^{1} g_d(s)\, (1 - s^2)^{(d-3)/2}\, ds. \qquad (A.9)_d$$

It is an elementary fact that for every (ξ, η)

$$-1 < s < 1 \quad \text{implies} \quad \frac{1}{\xi^2 + \eta^2 - 2\xi\eta s} \leqslant \frac{1}{(1-s^2)\,\eta^2}. \tag{A.10}$$

Insertion of this inequality in definition $(A.8)_d$ yields

$$g_d(s) \leqslant \left(\frac{1}{1-s^2}\right)^{(d-2)/2} \left(\frac{1}{\eta}\right)^{d-2}.$$

This, in turn, inserted in relation $(A.9)_d$ yields

$$\int_{\mathscr{S}(u,\alpha)} \left(\frac{1}{|\xi u - \eta v|}\right)^{d-2} dS_v \leqslant (d-1)\,\sigma_{d-1} \int_{\cos\alpha}^{1} \left(\frac{1}{1-s^2}\right)^{1/2} ds \cdot \left(\frac{1}{\eta}\right)^{d-2} \tag{A.11$_d$}$$

According to definition (A.6)

$$1 - \cos\alpha = \begin{cases} \dfrac{\sqrt{1 + \left(\dfrac{\delta}{\xi - \delta}\right)^2} - 1}{\sqrt{1 + \left(\dfrac{\delta}{\xi - \delta}\right)^2}} & \xi > \delta \\ 2 & \xi < \delta. \end{cases}$$

Hence

$$1 - \cos\alpha = O(1) \min\left(1, \left(\frac{\delta}{\xi - \delta}\right)^2\right),$$

and

$$\int_{\cos\alpha}^{1} \left(\frac{1}{1-s^2}\right)^{1/2} ds = O(1) \min\left(1, \left|\frac{\delta}{\xi - \delta}\right|\right). \tag{A.12}$$

Insertion of this estimate and of $(A.11)_d$ in inequality $(A.7)_d$ yields the validity of conclusion $(A.1)_d$.

To verify conclusion $(A.1)_2$ set

$$f_2(v) = \log\left(\frac{1}{|\xi u - \eta v|}\right)$$

and

$$g_2(s) = \log \frac{1}{\xi^2 + \eta^2 - 2\xi\eta s}. \tag{A.8$_2$}$$

Then the application of Lemma A yields

$$\int_{\mathscr{S}(u,\alpha)} \log \frac{1}{|\xi u - \eta v|}\, ds_v = \sigma_1 \int_{\cos\alpha}^{1} g_2(s)\,(1 - s^2)^{-1/2}\, ds. \tag{A.9$_2$}$$

Inserting inequality (A.10) in definition $(A.8)_2$ we obtain

$$g_2(s) \leqslant \log \left(\frac{1}{1 - s^2}\right) + 2 \mid \log \eta \mid .$$

An adaptation of the arguments leading to estimate (A.12) yields

$$\int_{\cos\alpha} \log \left(\frac{1}{1 - s^2}\right) (1 - s^2)^{-1/2} \, ds$$

$$= O(1) \min \left(1, \left(1 + \left|\log \left|\frac{\delta}{\xi - \delta}\right|\right|\right) \left|\frac{\delta}{\xi - \delta}\right|\right).$$

These three inequalities together show that

$$\int_{\mathscr{S}(u,\alpha)} \log \left(\frac{1}{\mid \xi u - \eta v \mid}\right) dS_v$$

$$= O(1) \min \left(1, \left(1 + \left|\log \left|\frac{\delta}{\xi - \delta}\right|\right|\right) \left|\frac{\delta}{\xi - \delta}\right|\right) (1 + \mid \log \eta \mid).$$

Inserting this inequality in inequality $(A.7)_2$ we arrive at the validity of estimate $(A.1)_1$.

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